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THE CATENARY

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A catenary is the shape taken by a uniform chain or string freely suspended from two points. The parameters of this shape for a suspended chain are measured and then compared to the mathematical predictions.

I. INTRODUCTION

The shape of a flexible chain with a uniform mass distribution which is freely suspended under gravity from two points on a horizontal line is described by[1,2]

$$y(x) = a\cosh(x/a) + b, \tag{1}$$

where a is the ratio of the tension at the midpoint of the cable to its weight per unit length, and b depends on a as well as on the choice of the origin for the coordinate system. The curve described by Eq.(1) is called the catenary. We will use a coordinate system where the two suspension points are symmetrically located at $x = d$ and $x = -d$. Consider the situation where the $y = 0$ is along the horizontal line defined by the two suspension points. **Show that** $b = -a\cosh(d/a)$. If $y = 0$ is defined as the lowest point of the catenary, then **show that** for this case, $b = -a$ and that Eq.(1) becomes

$$y(x) = a\cosh(x/a) - a. \tag{2}$$

II. PROCEDURE

A fine, flexible chain and large sheets of centimeter graph paper are provided for this experiment. Use the chain as a plumb-bob to precisely align the y-axis of the graph paper on the cork-board with the gravitational vertical. Suspend the chain from two pins along a horizontal line. Check to see that the chain has no kinks or turns in it. It would be smart to chose the suspension points on major divisions of the graph paper so that you have a symmetrically located vertical line. Tap the cork board a few times so that the shape of the chain is not altered by contact with the paper. It may be helpful to use a pair of additional pins above the two suspension points so that the shape taken by the chain is not perturbed by the two pins that define the suspension points. Record the shape of the chain by puncturing the paper with a pin along the curve of the chain. Do this for about twenty or so points. (It will make life a little easier later if you do this for pairs of y values located at the same $|x|$ value.) Do a careful job of locating the lowest point on the catenary. You may want to use a pencil to draw a small circle around each pin hole, including the suspension points, for ease in locating the holes later. Mark the two suspension points on the chain before your remove the chain from cork-board so that you can record the length of the chain between these two points.

III. DATA ANALYSIS

Digitize the experimental points and tabulate them. Define the minimum value of y as $y = 0$ and rescale the y coordinates if necessary. The idea here is to deduce the value of the parameter a and then to compare the measured shape to that of Eq.(2). A number of different methods to find the value of a are discussed in the following paragraphs. You should try them all.

If you do a simple arc length integral along the path of the chain[1], then the length of the chain can be expressed as

$$L = 2a \sinh(d/a), \tag{3}$$

where $2d$ is the horizontal distance between the suspension points. Since you know both L and d, Eq.(3) is just one equation in one unknown. However, the equation is a transcendental equation and has to be solved graphically or numerically. The solution is easier if we rewrite Eq.(3) in a more convenient form. Defining a variable $Z = L/(2a)$ and a fixed parameter $p = 2d/L$, Eq.(3) becomes

$$f(Z) = \sinh(pZ) - Z = 0. \tag{4}$$

We want to consider a numerical method to solve this equation. The Newton-Rapson algorithm for finding the root x to the equation $f(x) = 0$ is

(0) Guess or estimate a value for the root and call it x_0.

(1) Calculate $x_1 = x_0 - f(x_0)/f'(x_0)$. The value of x_1 is a better estimate of the root than is x_0.

(2) Calculate $x_2 = x_1 - f(x_1)/f'(x_1)$. The value of x_2 is an even better estimate.

(n+1) Calculate $x_{n+1} = x_n - f(x_n)/f'(x_n)$. The value of x_{n+1} is better yet.

Your root is accurate to at least s significant digits if the $s+1$ digit doesn't change between the n and $n+1$ iteration. Either use a built-in root finding function in Mathcad (or similar package), or write a computer program to use this method to find the root of $f(Z) = \sinh(pZ) - Z = 0$ to at least a four-digit accuracy. Determine the uncertainty in Z due to the uncertainty in p. Then deduce a value for a and its uncertainty.

Another approach would be to use the distance $h = y(d) - y(0)$ from your data. Thus $h = a\cosh(d/a) - a$ gives you an equation to solve for a. Define a variable $Q = h/a$ and a fixed parameter $r = d/h$ and then find the root of $\cosh(rQ) - Q - 1 = 0$. What is the uncertainty associated with Q? Then find a and its uncertainty. Calculate a value of L and compare it to the measured value.

In order to quantitatively discuss how well a model curve fits the data points, calculate the standard deviation,

$$\sigma = \sqrt{\frac{1}{N}\sum_i (y_i - y(x_i))^2}, \tag{5}$$

This quantity is the root-mean-square deviation of y_i from $y(x_i)$. The best value of a is the one which minimizes the value of σ. (It is possible to create a short Mathcad worksheet or computer program to calculate σ using your data set and a given value of a. It is then easy to find which value of a yields the smallest value of σ.)

Simple polynomials are usually easier to work with are transcendental functions like $\cosh(x)$. With this in mind, do a Taylor series expansion of $y(x) = a\cosh(x/a) - a$ about the point $x = 0$ and show $y(x) \approx x^2/2a$. Use this to estimate a for a few points near $x = 0$.

IV. PRESENTATION OF RESULTS

You should tabulate the following quantities; x_i, y_i and $y(x_i)$. Plot your data points and the curve $y(x) = a\cosh(x/a) - a$. Tabulate and plot the difference between y_i and $y(x_i)$. Look for any systematic errors. Overlay the $y(x) \approx x^2/2a$ approximation on your data points and comment on where this approximation is a good representation of the catenary.

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[1] Keith R. Symon, Mechanics, Third ed. (Addison-Wesley, Reading MA, 1971), pp. 237-241.

[2] Paul Kunkel, `http://whistleralley.com/hanging/hanging.htm`