Damped Driven Harmonic Oscillator and Linear Response Theory

Physics 258-259
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Purpose:

1. To measure and analyze the response of a mechanical damped harmonic oscillator. Both the impulse response and the response to a sinusoidal driving force are to be measured.

2. Using linear response theory, analyze the impulse response to predict the frequency-dependent response to sinusoidal excitation. The results can be compared both with theory and with the direct experimental measurements obtained with sinusoidal excitation.

Equipment:

1. Bifilar leaf-spring oscillator with Hall-effect position sensor and magnetic coil drivers for excitation and damping (or an equivalent oscillator/sensor combination).
2. National Instruments interface board and the LabWindows program `DDHO.exe` to drive it.
3. Micrometer and mount, for calibrating response of the position sensor.
4. MathCad example programs regarding impulse response and least-squares fitting of a damped sinusoidal oscillation.

Theory

The basic theory of a damped harmonic oscillator is given in detail in most introductory physics textbooks. If we assume that the damping force is proportional to velocity (actually a somewhat arbitrary assumption for a mechanical oscillator, but a reasonable one), the equation of motion for a harmonic oscillator is,

$$m\ddot{x} + b\dot{x} + kx = 0\,. \tag{1}$$

Define the free-running frequency as usual,

$$\omega_0 = \sqrt{\frac{k}{m}}\,. \tag{2}$$

We will assume that the oscillator is under-damped, so that oscillatory solutions exist. The condition for this to be true is

$$b < 2m\omega_0\,. \tag{3}$$

A. Transient Response

If there is no driving term after t=0, the response can be found using standard methods for ordinary differential equations, or alternatively by substituting a guess in the form of a damped sinusoidal oscillation. In complex notation this guess takes the form,

$$\hat{x} = \hat{C}e^{i\hat{\omega}t}. \tag{4}$$

Substituting and solving, then taking the real part, the solution for under-damped motion is,

$$x = Ae^{\frac{-bt}{2m}}\cos(\omega' t + \phi), \text{ with } \omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \tag{5}$$

This result shows that the resonance frequency is shifted slightly due to the damping. The arbitrary constants A and ϕ must be determined from the initial conditions.

B. Steady-State Response to Sinusoidal Excitation

If the harmonic wave is driven by a sinusoidal force at frequency w with constant amplitude,

$$F = D\cos(\omega t), \tag{6}$$

the steady-state response after transient motion has died out is given by

$$\begin{aligned} &x = x_0\cos(\omega t + \theta), \text{ where} \\ &x_0 = \frac{D}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}, \text{ and} \\ &\theta = \tan^{-1}\frac{-b\omega}{m(\omega_0^2 - \omega^2)} \end{aligned} \tag{7}$$

Near resonance, the amplitude becomes large but remains finite, while the phase approaches $-\pi/2$ (meaning that the velocity is in phase with the applied force). Plots of the amplitude and phase as a function of frequency appear in the MathCad supplement on linear response theory and analysis of the impulse response. Although obtained by a different method, they are identical to Eq. (7).

Note that this result applies only for sinusoidal excitation that has been present for a long time (much longer than the damping time constant). The general response of the harmonic oscillator to arbitrary excitation is a sum of transient and steady-state solutions. A powerful method for quantitatively predicting the general solution is presented below.

C. Impulse Response and the Transfer Function

A different and very powerful perspective on the behavior of linear systems can be obtained from the superposition principle. The idea of a *transfer function* is much-used in engineering, but is somewhat underappreciated in physics because it is basically a special case of the broader but more abstract idea of a *Green's function*, which we will not discuss here.

First consider the response of the damped harmonic oscillator to an impulsive excitation; that is, a pulse of very short time duration that nevertheless transfers a finite energy to the oscillator. The *impulse response h(t)* is defined to be the response (in this case the time-varying position) of the system to an impulse of unit area. For the harmonic oscillator, its form is given by Eq. (5).

The key concept is that because the equation of motion is linear, solutions can be superposed. In particular, the response to a succession of impulses is just the sum of the responses to the separate impulses. It then follows that, because *any* function can be approximated as a series of impulses with varying amplitudes, we can calculate the response to *any* time-dependent driving term if we know the impulse response! This can be expressed mathematically for an arbitrary driving term *D(t)* by integration over a series of infinitesimal impulse responses of size $h(\tau)D(t-\tau)$, yielding the total response $x(t)$:

$$x(t) = \int_0^\infty h(\tau)D(t-\tau)d\tau = \int_{-\infty}^t D(\xi)h(t-\xi)d\xi \,. \tag{8}$$

If we take care that $h(t)=0$ for negative times, this can be written in the standard form of a *convolution* integral,

$$x(t) = \int_{-\infty}^\infty D(\xi)h(t-\xi)d\xi \tag{9}$$

All of this can be re-expressed using the language of Fourier transforms, in a way that is both computationally straightforward and intellectually rich. Most computer languages have a pre-programmed *fast Fourier transform*, or FFT, which makes short work of the required numerical operations. We perform a Fourier transform on the impulse response to obtain the frequency-domain *transfer function*,

$$T(\omega) = \tilde{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty h(t)e^{i\omega t}dt \tag{10}$$

Its physical interpretation is extremely simple. It describes the response to an impulse (or delta-function) excitation that has equal components at all frequencies. As a result, the transfer function at frequency ω directly describes the response of the oscillator to a sinusoidal driving term at frequency ω. Its magnitude describes the amplitude response, and its phase describes the phase shift. In short, it is *exactly* the same as the familiar plot of the amplitude and phase response of a harmonic oscillator as a function of the driving frequency!

The *convolution theorem* of mathematics tells us that in Fourier transform language, the time-domain convolution integral of Eq. (9) corresponds to simple multiplication in the frequency domain. More specifically we find the frequency-domain response of the harmonic oscillator to an arbitrary driving force $D(t)$ by calculating the Fourier transform $\tilde{D}(\omega)$ of the driving force, then multiplying it by the transfer function,

$$\tilde{x}(\omega) = \tilde{h}(\omega) \cdot \tilde{D}(\omega) \tag{11}$$

The ordinary time-domain response $x(t)$ is obtained from this by performing an inverse Fourier transform back to the time domain,

$$x(t) = \mathfrak{F}(\tilde{x}(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{-i\omega t} d\omega \tag{12}$$

A separate MathCad spreadsheet available on the Physics 258 Web page, `Impulse_Response.mcd`, demonstrates this remarkable result for a few examples. It clearly verifies that if the impulse response is known, the response to any other excitation can be straightforwardly calculated.

Experiment

A. Calibration and Linearization of the Position Sensor

1. Using the computer interface, read the static voltage from the Hall sensor that corresponds to the equilibrium position of the oscillator.

2. Use a micrometer to mechanically translate the oscillator mass first to one side, then the other, in calibrated increments. Record the Hall voltage corresponding to each position, extending for a range of several mm on each side of equilibrium. (*Question*: How can you easily tell when the micrometer first makes contact?).

3. Analyze the results using MathCad or MatLab to obtain a function *X(V)* that calculates the actual position of the oscillator mass, given the voltage measured by the Hall sensor. If it is significantly nonlinear, you can improve your results if you use this function to process all of your position data. (Hint: Although you can use interpolation of the measured data, you will probably obtain better results by fitting the data to a low-order polynomial, using polynomial regressions. *Question*: how many terms are enough?)

B. Observation and Analysis of the Impulse response

Using the LabWindows program `ddho.exe`, measure the impulse response to a sudden excitation. Use a large-amplitude driving pulse (perhaps 5 V) with a duration very short compared to the free-running period of the oscillator, no more than 0.1 seconds. You will probably have trouble using a pulse much shorter than 0.05 seconds, though, because the amplitude of the motion will be so small that the signals will be very noisy. The main effect of using a driving pulse with a small but finite width is to cause a small delay in the response, which has little effect in the FFT analysis of the amplitude, but is noticeable in the phase (see Section B.2). When taking data, make sure that your sampling period is long enough to allow for nearly complete damping, and that your sampling rate is fast enough to accurately represent the oscillations (at least 5-10 data points per period).

Repeat the measurement both for the case of minimal damping and for at least one value of higher damping, obtained by using a magnetic damper. (*Question*: The magnetic damper involves passing a piece of copper through a region with a position-varying magnetic field. How does it work?)

1. Use a generalized least-squares fitting program to fit the data to a damped exponential decay curve. You can do this yourself if you wish, but you will probably want to start with a pre-written MathCad spreadsheet available on the Physics 258 web page, `Damped_sine_fit.mcd`, so you don't have to go through the tedious task of re-typing the partial derivatives of the fitting function. To obtain the best

results at large amplitudes, consider applying the linearization function from part A before fitting the data. From the fits, determine the damping constants *b/2m* and the shifted frequencies ω'. Calculate the free-running frequency ω_0 from these results. How well do the results obtained using different damping constants agree with one another?

2. Transform the impulse response using a Fourier transform (i.e., find the transfer function), then use it to predict the amplitude and phase response of the oscillator to steady sinusoidal excitation, as a function of frequency. You will again probably want to make use of the examples provided in a pre-written MathCad spreadsheet, `Impulse_Response.mcd`. To avoid confusion that has arisen in past years, it is important to note that your results should be calculated graphs of the predicted frequency-dependent amplitude and phase—it's not sufficient just to show that if you Fourier transform the impulse response and then perform an inverse transform, you get back what you started with. How well does the functional form in your graphs agree with the theoretical expression in Eq. (7)? Do you see a slight linear slope in the phase plot that "shouldn't be there"? See if you can interpret this in terms of a delay due to the finite duration of the driving pulse. Can you obtain a similar effect in a numerical simulation by inserting a slight phase offset into the sample waveform provided in `Impulse_Response.mcd?` Finally, note that you can minimize this effect by discarding the first several points of the measured impulse response, choosing a new starting point in the data set so that it nearly coincides with a zero crossing.

C. Response to a sinusoidal driving force

Measure the response to sinusoidal excitation at a variety of frequencies near resonance, as well as a few frequencies further away. Be sure to record data for a sufficiently long time to include both the transient and the steady-state response to the driving force.

1. Analyze the *steady-state* amplitude of the response $x(t)$ at large times. Plot your results to obtain a resonance curve, showing the amplitude as a function of frequency for sinusoidal excitation. Also draw a phase plot, showing the relative phase between the response and the driving term, as a function of frequency.

2. Compare the center frequency and width of the resonance curve with the predictions of linear response theory, using the transfer function obtained from the impulse response.

3. Compare the shape of the amplitude response with the theoretical expression in Eq. (7). To facilitate this, note that in the limit of small damping, the amplitude $A(f)$ can be approximated by a Lorentzian with a full-width at half-maximum determined by the damping. Do your results (for small damping) agree with the predicted value using the measured damping term?

4. You can also predict the *short-time* response to a suddenly applied sine wave burst by using the transfer function. Calculate the predicted response for frequencies differing by about 5% from the resonant frequency, where you should be able to see beat notes between the applied frequency and the natural resonant frequency. Compare the predictions with your measured short-time position data.