Self-duality, helicity and higher-loop effective actions

Gerald Dunne

- *two-loop Euler-Heisenberg effective action in a self-dual background is remarkably simple*
- *two-loop = (one-loop)* ^{2} *+ one-loop*
- *background field "loopology"*
- *algebraic perspective on mass renormalization*

GD, C. Schubert, H.Gies : PLB, JHEP
GD: Ian Kogan Memorial Volume

Coral Gables, December 19, 2004

context

- dramatic recent progress in tree & multiloop amplitudes
- helicity and color decompositions, integration-by-parts, master diagrams, recurrence/differential relations, SUSY…
- twistor methods

Fundamental idea : expand about helicity amplitudes

SUGRA, Parkes-Taylor, Siegel, Bardeen, Nair, Cangemi, Bern-Dixon-Kosower, ...
Witten, Cachazo-Svrcek-Witten, Brandhuber et al, Bern et al, …

- all two-loop 2 --> 2 QCD processes
- two-loop gluon fusion : g g --> $\gamma\,\gamma$
- two-loop QED, QCD corrections to : $\bar{q}q$ --> $\gamma\gamma$
  $\bar{q}\,q$ --> g γ
  e^+e^- --> $\gamma\gamma$

all since 2000

basic strategy and context

• complementary approach: background field

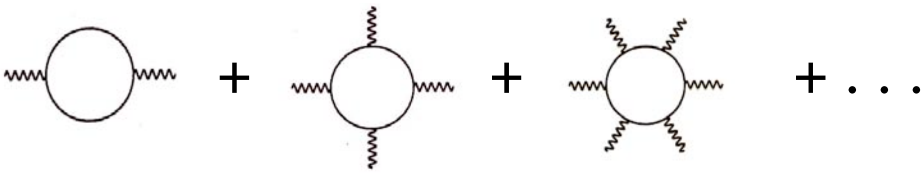
higher-loop effective Lagrangian =
generating function for higher-loop amplitudes

goal : develop higher-loop background field loop tools, a la amplitudes

perturb around self-duality -- signs of magic

One-loop effective action

$$S[A] \;=\; -\frac{i}{2}\log\det\left(\not{D}^2+m^2\right)$$



Heisenberg & Euler (1936) ; Weisskopf (1936)

$$\mathcal{L}^{(1)}=-\frac{1}{8\pi^2}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\left[\frac{e^2ab}{\tanh(eaT)\tan(ebT)}-\frac{1}{T^2}-\frac{e^2}{3}(a^2-b^2)\right]$$

constant field-strength $F_{\mu\nu}$

$$F=\left(\begin{array}{cc|cc} & a & & \\ -a & & & \\ \hline & & & b \\ & & -b & \end{array}\right)$$

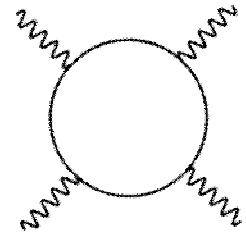
$$a\,b=\vec{E}\cdot\vec{B}$$

$$a^2-b^2=\vec{B}^2-\vec{E}^2$$

One-loop Applications

Light-light scattering (Euler & Kockel, 1936)

$$S = \frac{e^4}{360\pi^2 m^4} \int d^4x \left[(\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2\right] + \dots$$



Pair production from vacuum (Heisenberg & Euler, 1936)

$$Im\,\mathcal{L}^{(1)} \sim \frac{e^2E^2}{8\pi^3} \exp\left[-\frac{m^2\pi}{eE}\right]$$

β - function from strong field limit : (Weisskopf, 1936)

$$\mathcal{L}^{(1)} \sim \left(\frac{e^2}{12\pi^2}\right)\frac{B^2}{2}\ln\left(\frac{eB}{m^2}\right)$$

Beyond QED : QCD, SUSY, gravity, SUGRA, strings, …

Mass renormalization

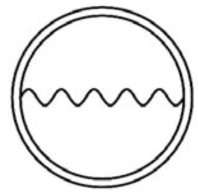
loop expansion : $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots$

$$= \mathcal{L}^{(0)}_{\text{ren}} + \mathcal{L}^{(1)}_{\text{ren}}(m^2) + \mathcal{L}^{(2)}_{\text{ren}}(m^2) + \dots$$

mass shift : $m^2 = m_0^2 + \delta m^2 + \dots$

mass renormalization : $\mathcal{L}^{(1)}_{\text{ren}}(m^2) = \mathcal{L}^{(1)}_{\text{ren}}(m_0^2) + \delta m^2 \frac{\partial \mathcal{L}^{(1)}_{\text{ren}}}{\partial m^2} + \dots$

Two-loop E-H effective actions



Propagator in constant background (Fock, Nambu, Schwinger)

$$G_{\rm scalar}(x,x') = -i\frac{e^{-i\eta}}{(4\pi)^2}\int_0^\infty \frac{dT}{T^2}\exp\left[-im^2T - L(T) + \frac{i}{4}z\beta(T)z\right]$$

$$\begin{aligned} z_\mu &\equiv x_\mu - x'_\mu \\ \beta_{\mu\nu} &\equiv [eF\coth(eFT)]_{\mu\nu} \\ L &\equiv \frac{1}{2}{\rm tr}\ln\left(\frac{\sinh(eFT)}{eFT}\right) \end{aligned}$$

Ritus (1975) : renormalized expression for $L^{(2)}$

Charge and mass renormalization

Complicated two-parameter integrals !

Ritus's result (simplified!)

$$\mathcal{L}^{(2)}_{\rm ren} = \alpha\,\frac{m^4}{(4\pi)^3}\left(\frac{eB}{m^2}\right)^2\int_0^\infty \frac{ds}{s^3}\,e^{-m^2 s/(eB)}\int_0^1 du\,\left[L(s,u) - 2s^2 + \frac{6}{u(1-u)}\left(\frac{s^2}{\sinh^2 s} + s\coth s\right)\right] -$$
$$12\alpha\,\frac{m^4}{(4\pi)^3}\frac{eB}{m^2}\int_0^\infty \frac{ds}{s}\,e^{-m^2 s/(eB)}\left[\coth s - \frac{1}{s} - \frac{s}{3}\right]\left[\frac{3}{2} - \gamma - \log\left(\frac{m^2 s}{eB}\right) + \frac{eB}{m^2 s}\right]$$

$$L(s,u) = s\coth s\left[\frac{log\left(\frac{u(1-u)}{G(u,s)}\right)}{[u(1-u)-G(u,s)]^2}F_1 + \frac{F_2}{G(u,s)[u(1-u)-G(u,s)]} + \frac{F_3}{u(1-u)[u(1-u)-G(u,s)]}\right]$$

$$G(u,s) = \frac{\cosh s - \cosh((1-2u)s)}{2s\sinh s}$$

$$F_1 = 4s(\coth s - \tanh s)G(u,s) - 4u(1-u)$$

$$F_2 = 2(1-2u)\frac{\sinh((1-2u)s)}{\sinh s} + s(8\tanh s - 4\coth s)G(u,s) - 2$$

$$F_3 = 4u(1-u) - 2(1-2u)\frac{\sinh((1-2u)s)}{\sinh s} - 4s\,G(u,s)\tanh s + 2$$

very complicated!!!

Two-loop : self-dual magic (GD, C. Schubert, 2002)

Self-dual background : $F_{\mu\nu} = \tilde{F}_{\mu\nu}$

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = f^2$$

$$\mathcal{L}^{(2)}_{\text{spinor}} = -\alpha^2 \frac{f^2}{2\pi^2} \left[3\, \xi^2(\kappa) - \xi'(\kappa)\right]$$

$$\mathcal{L}^{(2)}_{\text{scalar}} = \alpha^2 \frac{f^2}{4\pi^2} \left[\frac{3}{2} \xi^2(\kappa) - \xi'(\kappa)\right]$$

dimensionless parameter : $\kappa \equiv \frac{m^2}{2ef}$

Why so simple?

Why so similar?

ubiquitous function :

$$\xi(\kappa) \equiv -\kappa \left(\psi(\kappa) - \ln(\kappa) + \frac{1}{2\kappa}\right)$$

Why ξ and ξ' ?

Two-loop self-dual background

Why so simple?

Self-duality ⇒ $F_{\mu\nu} F_{\nu\rho} = - f^2 \delta_{\mu\rho}$ ⇒ propagators simplify

$$G_{\rm scalar}(x,x') = -i\frac{e^{-i\eta}}{(4\pi)^2}\int_0^\infty \frac{dT}{T^2}\exp\left[-im^2T - L(T) + \frac{i}{4}z\beta(T)z\right]$$

$$\begin{aligned} z_\mu &\equiv x_\mu - x'_\mu \\ \beta_{\mu\nu} &\equiv [eF\ \coth(eFT)]_{\mu\nu} \\ L &\equiv \frac{1}{2}\mathrm{tr}\ \ln\left(\frac{\sinh(eFT)}{eFT}\right) \end{aligned}$$

⇒

massive scalar $$G_{\rm scalar}(p) = \int_0^\infty \frac{dt}{\cosh^2(eft)}\, e^{-m^2t-\frac{p^2}{ef}\tanh(eft)}$$

massless scalar $$G_{\rm scalar}(p) = \frac{1-e^{-\frac{p^2}{ef}}}{p^2}$$

Two-loop self-dual background

Why so simple?

Self-duality ⇒ $F_{\mu\nu} F_{\nu\rho} = - f^2 \delta_{\mu\rho}$ ⇒ propagators simplify

Self-duality ⇒ definite helicity : $\sigma_{\mu\nu} F_{\mu\nu}(1+\gamma_5)/2 = 0$

(massless) helicity amplitudes in QED, QCD, gravity, SUSY, SUGRA known to be simple

Why so similar ?

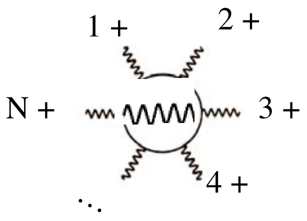
Self-duality ⇒ Dirac operator has QM supersymmetry

't Hooft
Jackiw, Rebbi

one-loop : $\mathcal{L}^{(1)}_{\rm spinor} = -2\mathcal{L}^{(1)}_{\rm scalar} + \frac{1}{2}\left(\frac{ef}{2\pi}\right)^2 \ln\left(\frac{m^2}{\mu^2}\right)$

two-loop : helicity projections of propagators

Application : two-loop helicity amplitudes



SD effective action = generating function of "all +" helicity amplitudes

$$\Gamma^{(2)}[k_1,\epsilon_1^+;k_2,\epsilon_2^+;\ldots;k_N,\epsilon_N^+] \;=\; \alpha\pi\frac{(2e)^N}{(4\pi)2m^{2N-4}}\,c^{(2)}_{N/2}\,\chi_N$$

spinor helicity elements

GD, C.Schubert, JHEP 2003

$$\chi_N \equiv \frac{(N/2)!}{2^{N/2}}\left\{[12]^2[34]^2\cdots[(N-1)N]^2 + \text{all perms}\right\}$$

expansion coefficients

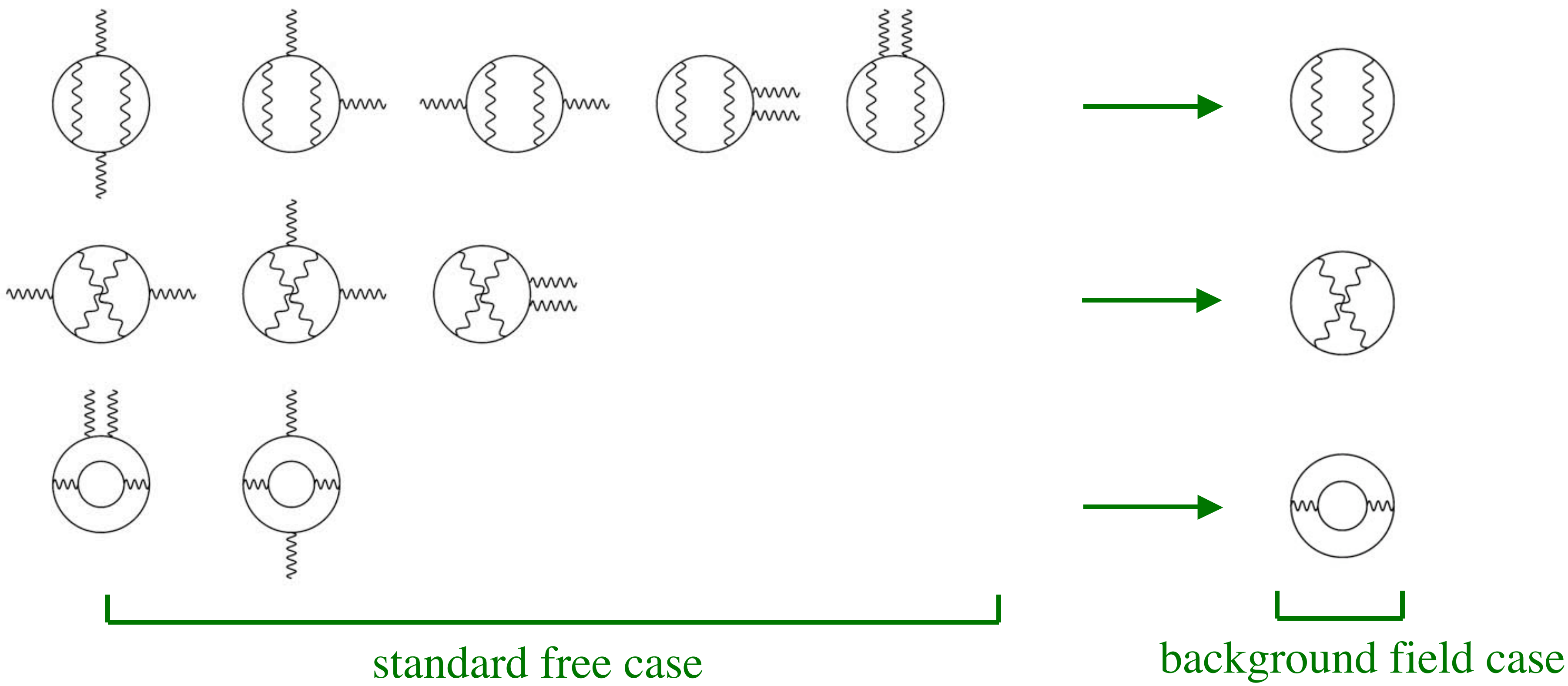
$$c^{(2)}_n \;=\; \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\,\mathcal{B}_{2n-2} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{(2n-2k)}\right\}$$

any number of legs ; but low-energy limit

generalize to any helicity combinations (hep-th/0301022)

Effective Lagrangian and β - function

Advantages : combinatorics and graph structure   e.g. 3-loop



Disadvantage : need background field propagators

BUT : self-dual propagators are almost as simple as free ones

Strong-field limit and β - function

scale anomaly : $\langle \Theta^{\mu\nu} \rangle = -\eta^{\mu\nu}\mathcal{L} + 2\frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}}$ $\quad$ $\langle \Theta^{\mu}_{\mu} \rangle = \frac{\beta(\bar{e})}{2\bar{e}} \frac{e^2}{\bar{e}^2} (F_{\mu\nu})^2$

$$\boxed{\mathcal{L} = -\frac{1}{4}\frac{e^2}{\bar{e}^2(t)} F_{\mu\nu}F^{\mu\nu}}$$

$$\beta(\bar{e}(t)) \equiv \frac{d\bar{e}(t)}{dt}$$

$$t \equiv \frac{1}{4} \ln\left(\frac{e^2|F^2|}{\mu_0^4}\right)$$

perturbatively: $\quad t = \int_e^{\bar{e}(t)} \frac{de'}{\beta(e')} \quad , \quad \beta(e) = \beta_1 e^3 + \beta_2 e^5 + \ldots$

$$\frac{1}{\bar{e}^2(t)} = \frac{1}{e^2} - 2\beta_1 t - 2\beta_2 e^2 t + O(e^4 t^2)$$

$$\boxed{\mathcal{L} = \frac{1}{16}\left(2\beta_1 e^2 + 2\beta_2 e^4 + \ldots\right) F_{\mu\nu}F^{\mu\nu} \ln\left(\frac{e^2|F^2|}{\mu_0^4}\right)}$$

(Cf. Gell-Mann-Low)

note : assumes well-defined massless limit

Effective Lagrangian and β - function

Idea : self-dual background appears to be the 'simplest'

$$\mathcal{L}^{(1)}_{\rm scalar} \sim \frac{e^2}{48\pi^2} f^2 \ln\left(\frac{ef}{m^2}\right) \qquad \mathcal{L}^{(1)}_{\rm spinor} \sim -\frac{e^2}{24\pi^2} f^2 \ln\left(\frac{ef}{m^2}\right)$$

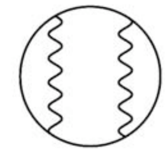
$$\mathcal{L}^{(2)}_{\rm scalar} \sim \frac{e^4}{64\pi^4} f^2 \ln\left(\frac{ef}{m^2}\right) \qquad \mathcal{L}^{(2)}_{\rm spinor} \sim -\frac{e^4}{32\pi^4} f^2 \ln\left(\frac{ef}{m^2}\right)$$

$$\beta_{\rm scalar} = \frac{e^3}{48\pi^2} + \frac{e^5}{64\pi^4} + \dots \qquad \beta_{\rm spinor} = \frac{e^3}{12\pi^2} + \frac{e^5}{64\pi^4} + \dots$$

- mis-match for spinor QED due to fermion zero modes

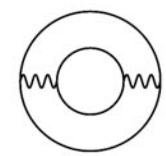
$$\mathcal{L}^{(1)}_{\rm spinor} = -2\mathcal{L}^{(1)}_{\rm scalar} + \frac{1}{2} N_0 \ln\left(\frac{m^2}{\mu^2}\right) \qquad N_0 = \left(\frac{ef}{2\pi}\right)^2$$

subtleties at > 1-loop for self-dual background (zero modes) (GD, Schubert, Gies, 2003)



Effective Lagrangians beyond two loops ?

- recursion for higher loops ?
- n-loop $\Longrightarrow$ (2 n - 2) parameter integrals (n > 1)



- background field “ loop-ology”
- key question : why ξ ?

At 2 loop: why ξ(κ) ?

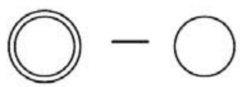
ξ(κ) = −κ(ψ(κ) − lnκ + 1/2κ)

κ=m²/(2ef)

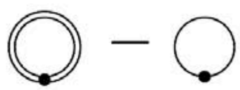
spinor: $$\mathcal{L}^{(2)}_{\rm spinor} = -6e^2\left(\frac{m^2\xi(\kappa)}{(4\pi)^2\kappa}\right)^2 + e^2\frac{(ef)^2}{2\pi^2}\frac{\xi'(\kappa)}{(4\pi)^2}$$

scalar: $$\mathcal{L}^{(2)}_{\rm scalar} = \frac{3e^2}{2}\left(\frac{m^2\xi(\kappa)}{(4\pi)^2\kappa}\right)^2 - e^2\frac{(ef)^2}{4\pi^2}\frac{\xi'(\kappa)}{(4\pi)^2}$$

NOTE:



$$\equiv \int\frac{d^4p}{(2\pi)^4}\left[G(p)-G_0(p)\right] = -\frac{m^2}{(4\pi)^2}\frac{\xi(\kappa)}{\kappa}$$



$$\equiv \int\frac{d^4p}{(2\pi)^4}\left[(G(p))^2-(G_0(p))^2\right] = \frac{\xi'(\kappa)}{(4\pi)^2}$$

$$G_{\rm scalar}(p) = \int_0^\infty \frac{dt}{\cosh^2(eft)}\, e^{-m^2t-\frac{p^2}{ef}\tanh(eft)}$$

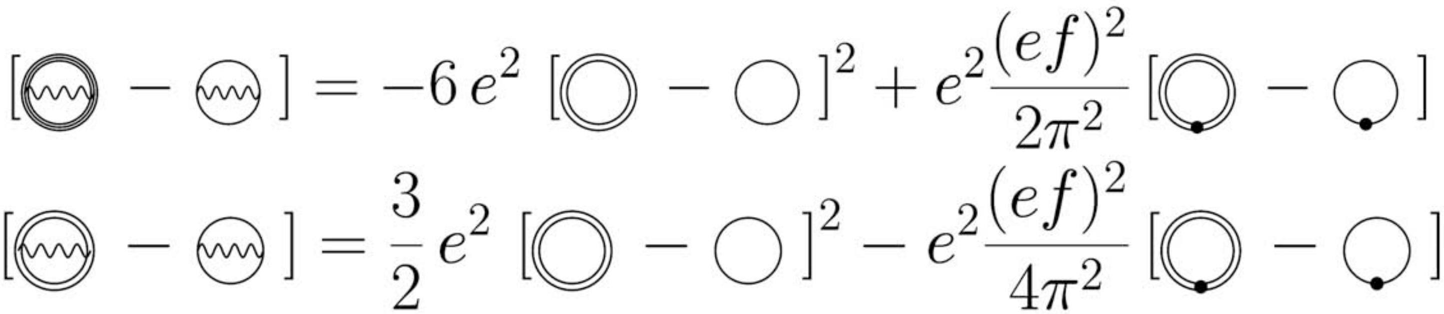
At 2 loop: why $\xi(\kappa)$?

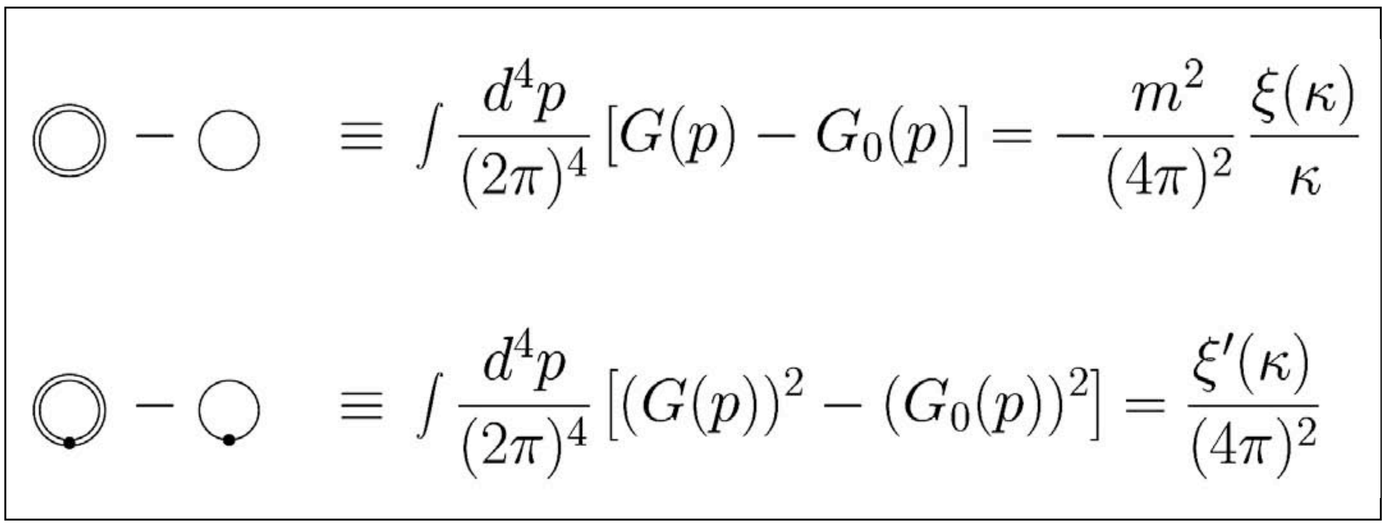
$$\xi(\kappa) = -\kappa(\psi(\kappa) - \ln\kappa + 1/2\kappa)$$

$$\kappa = m^2/(2ef)$$

spinor:

scalar:





two - loop = (one - loop)^2 + one - loop

this is very surprising!

4d ϕ^4 theory

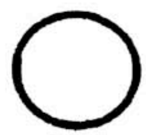
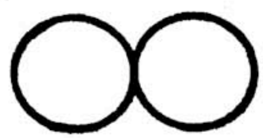
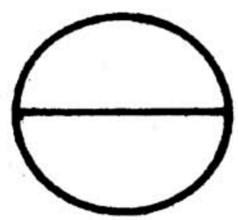
at two-loop

Iliopoulos, Itzykson & Martin, 1975

TABLE I. Contributions of the various diagrams to the value of the effective potential up to the order of two loops. The potential is written as

$$V(\varphi_c) = \frac{\mu^4}{\lambda}\mathcal{V}\left(\frac{\lambda\varphi_c^2}{2\mu^2}, \frac{\hbar\lambda}{(4\pi)^2}\right)$$

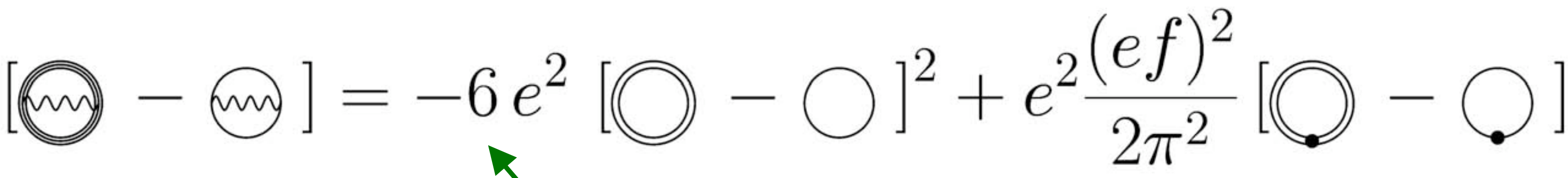
and x and α stand for $x = \lambda\varphi_c^2/2\mu^2$, $\alpha = \hbar\lambda/(4\pi)^2$.

	Diagram	Contribution
		x
$\mathcal{V}_0$		$\frac{x^2}{6}$
$\mathcal{V}_1$		$(\alpha/4)\left((1+x)^2\log(1+x) - (x+\frac{3}{2}x^2)\right)$
	 (a)	$(\alpha^2/8)\left((1+x)\log(1+x) - x\right)^2$
$\mathcal{V}_2$	 (b)	$\frac{1}{2}\alpha^2 x\left(\frac{1}{2}(1+x)\log^2(1+x) - 2(1+x) \times\log(1+x) + 2x\right)$

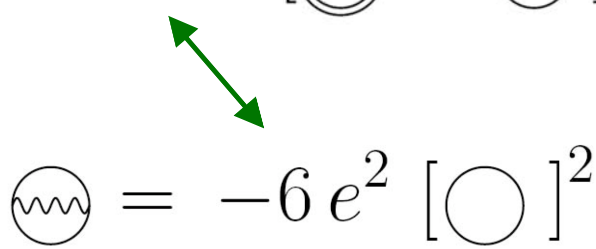
two - loop = (one - loop)^2 + one - loop

first hint : relation to diagrams with no background field

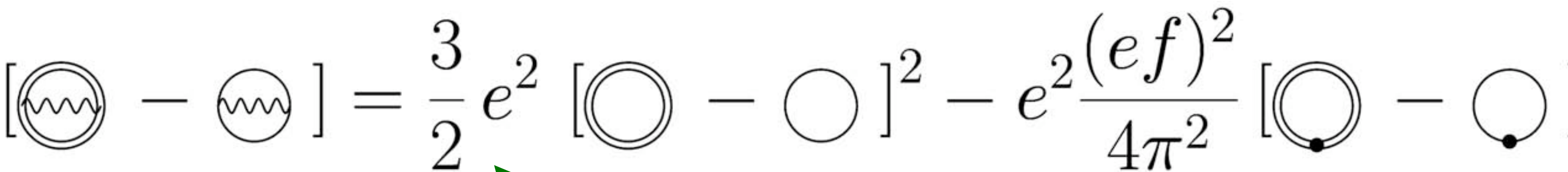
spinor:



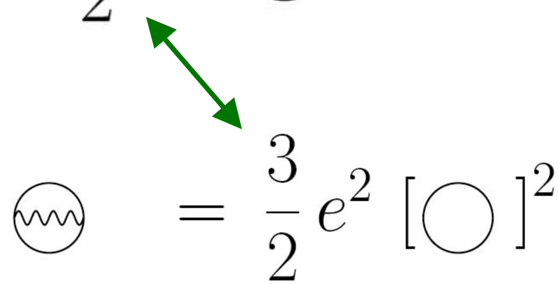
free spinor QED



scalar:



free scalar QED



background field "loopology"

GD, JHEP 2004

free scalar QED

$$
\bigcirc\!\!\!\!\sim \;=\; \frac{e^2}{2}\int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{(p+q)^2}{(p-q)^2(p^2+m^2)(q^2+m^2)}
$$

$$
= \frac{e^2}{2}\int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{\left[-(p-q)^2 + \cancel{2(p^2+m^2)} + \cancel{2(q^2+m^2)} - 4m^2\right]}{(p-q)^2(p^2+m^2)(q^2+m^2)}
$$

$$
= \frac{e^2}{2}\left(\frac{d-1}{d-3}\right)\left[\bigcirc\right]^2
$$

$$
0 = \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{\partial}{\partial p_\mu}\left[\frac{(p-q)_\mu}{(p-q)^2(p^2+m^2)(q^2+m^2)}\right]
$$

$$
= (d-2)\left[\ominus\right] - \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{2p\cdot(p-q)}{(p-q)^2(p^2+m^2)^2(q^2+m^2)}
$$

"integration-by-parts"

$$
= (d-2)\left[\ominus\right] - \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{\left[(p-q)^2 + (p^2+m^2) - (q^2+m^2)\right]}{(p-q)^2(p^2+m^2)^2(q^2+m^2)}
$$

$$
= (d-3)\left[\ominus\right] - \left[\bigcirc_{\bullet}\right]\left[\bigcirc\right]
$$

$$
= (d-3)\left[\ominus\right] - \frac{(d-2)}{2m^2}\left[\bigcirc\right]^2
$$

background field “loopology”

GD, JHEP 2004

self-dual background field scalar QED

$$= \frac{e^2}{2}\int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{1}{(p-q)^2}\left\{(p+q)^2 G(p)G(q) - (ef)^2 \frac{\partial G(p)}{\partial p_\mu}\frac{\partial G(q)}{\partial q_\mu}\right\}$$

$$= \frac{e^2}{2}\left(\frac{d-1}{d-3}\right)\left[\ \bigcirc\ \right]^2 + O(f^2) \ \text{ stuff}$$

$$A_\mu = -\frac{i}{2} F_{\mu\nu} \frac{\partial}{\partial p_\nu}$$

Background field “integration-by-parts”

pure algebra - no integrals !!!

$$(p^2+m^2)G(p) = 1 + \left(\frac{ef}{2}\right)^2 \frac{\partial^2 G(p)}{\partial p_\mu^2}$$

d-dimensional propagator

$$G_{\text{scalar}}(x, x') = -i\frac{e^{-i\eta}}{(4\pi)^2}\int_0^\infty \frac{dT}{T^2}\,\exp\left[-im^2T - L(T) + \frac{i}{4}z\beta(T)z\right]$$

$$z_\mu \equiv x_\mu - x'_\mu$$

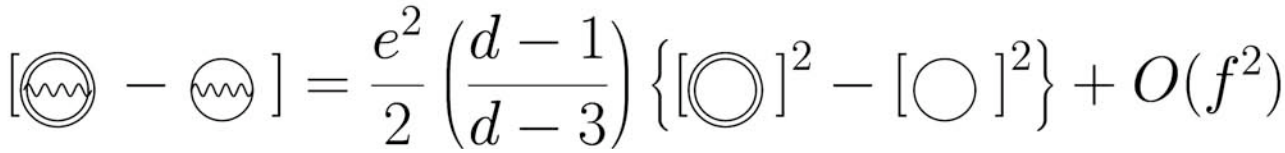
$$\beta_{\mu\nu} \equiv [eF\,\coth(eFT)]_{\mu\nu}$$

$$L \equiv \frac{1}{2}\text{tr}\,\ln\left(\frac{\sinh(eFT)}{eFT}\right)$$

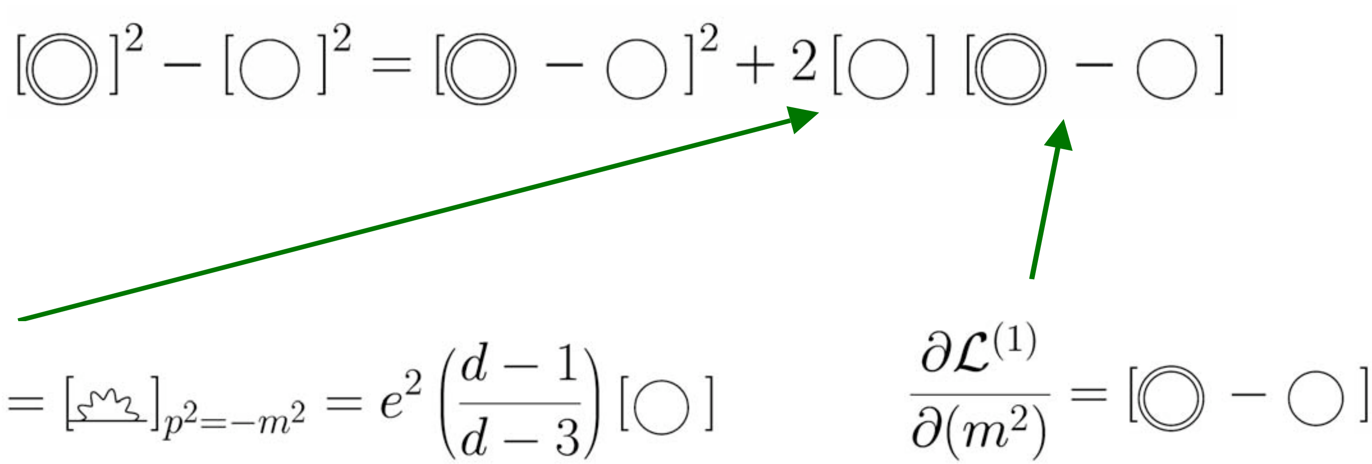
tr ln $\Rightarrow$ ln det

$$G(p) = \int_0^\infty \frac{dt}{\cosh^{d/2}(eft)}\,e^{-m^2t - \frac{p^2}{ef}\tanh(eft)}$$

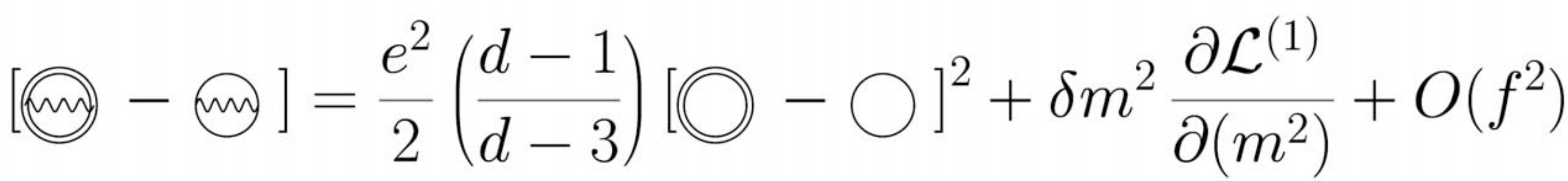
background field “loopology”: mass renormalization



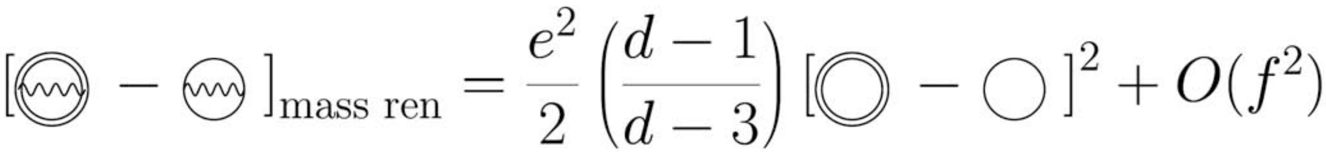
complete the square :



mass renormalization is a trivial algebraic loop operation



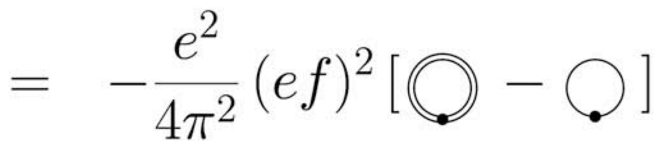
background field "loopology": charge renomalization



found the (one-loop)^2 part purely algebraically !

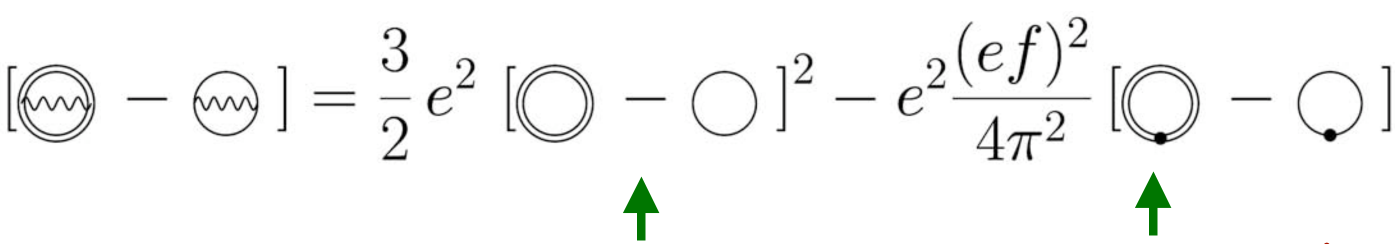
only remaining divergence is prop. to f ^{2} : charge renormalization

$$O(f^4) \xrightarrow{d\to 4} -4\pi^2 e^2 (ef)^2 \int \frac{d^4p\, d^4q}{(2\pi)^8} \left[G(p)G(q) - G_0(p)G_0(q)\right] \delta(p-q)$$



no integrals !!!

renormalized 2-loop effective Lagrangian :



generic !

non-generic

other connections

4d N = 4 SYM :

two-loop planar amplitudes related to squares of one-loop amplitudes

(Anastasiou, Bern, Dixon, Kosower, 2003)

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon)\, M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4$$

suggests possibility of iterative perturbative structure …

other connections

4d N = 2 SUSY QED : 2-loop HE effective Lagrangian

(Kuzenko & McArthur, 2004)

$$\mathcal{L}(F_+^2, F_-^2) = \Lambda(F_+^2) + \bar{\Lambda}(F_-^2) + F_+^2 F_-^2 \, \Omega(F_+^2, F_-^2)$$

'relaxed' self-duality condition :

$$\Psi^2 = \frac{1}{4}\bar{D}^2 \left(\frac{\bar{W}^2}{\bar{\Phi}^2 \Phi^2} \right) \quad , \quad \bar{\Psi}^2 = \frac{1}{4} D^2 \left(\frac{W^2}{\bar{\Phi}^2 \Phi^2} \right)$$

two-loop : $$\Gamma^{(2)}[W, \Phi] = \int d^8 z \frac{W^2 \bar{W}^2}{\bar{\Phi}^2 \Phi^2} \Omega^{(2)}(\Psi^2, \bar{\Psi}^2)$$

$$\Omega^{(2)}(\Psi^2, 0) = \frac{e^2}{2(4\pi)^4} - \frac{e^2}{4(4\pi)^4 \Psi^2} \xi'''(x)|_{x=1/\Psi}$$

similar result for N = 4 SYM

(Kuzenko & McArthur, 2004)

same function ξ

Conclusions

- self-dual background is a remarkably efficient probe
- physically : self-duality $\Leftrightarrow$ helicity $\Leftrightarrow$ QM SUSY
- renormalized 2-loop effective Lagrangians have very simple form
- loop recurrences : two - loop $=$ (one - loop) ^{2} $+$ (one - loop)
- simple graphical interpretation of mass renormalization

  - algebraic background field integration-by-parts rules
  - higher loops
  - nonabelian rules
  - Hopf algebra structure, a la Kreimer-Connes ?

Number theory application : Bernoulli identities

coth ^{2} x = 1- d(coth x)/dx

- Euler-Ramanujan :

$$\sum_{k=1}^{n-1} \binom{2n}{2k} \mathcal{B}_{2k}\mathcal{B}_{2n-2k} = -(2n+1)\mathcal{B}_{2n}$$

- Miki's identity (1978) : 'exotic', proved with p-adic analysis

$$\sum_{k=1}^{n-1} \frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)(2n-2k)} = \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)(2n-2k)} + \frac{\mathcal{B}_{2n}}{n}\left(\psi(2n+1)+\gamma\right)$$

- simple quantum field theory proof and generalizations:

$$\frac{\xi(\kappa)}{\kappa} = -\sum_{n=1}^{\infty} \frac{\mathcal{B}_{2n}}{2n} \frac{1}{\kappa^{2n}}$$

$$= \int_0^\infty ds\, e^{-2\kappa s}\left(\coth s - \frac{1}{s}\right)$$

GD, Schubert, 2004