MARKOVIAN DYNAMICS OF ULTRA-HIGH Q QUANTUM CAVITIES

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Abstract. We derive the Markovian master equation and the Jaynes-Cummings (JC) Hamiltonian for the ultra-high Q quantum optical cavity. It is shown that there is a correlation between the JC interaction model and the Markovian Lindblad master equation through the Liouville-von Neumann equation on the W^* algebra. First, the Jaynes-Cummings interaction potential is shown to arise from the coupling between quantized electromagnetic modes and the dipole moment of a simple quantum rigid rotor. Using this JC interaction potential, the two state JC Model Hamiltonian is derived. Also, using quantum stochastics and a first order Markovian approximation, the Quantum Langevin equation is derived from the Heisenberg equations of motion. The formalism of Completely Positive (CP) Quantum Semigroups is defined on the W^* algebra, and the origin of the Lindblad form of Markovian master equations is shown. This is done by the creation of a completely dissipative (CD) operator and by corollary it is shown that the Quantum Semigroup is CP iff the CD operator is of the Lindblad form. Using the Liouville-von Neumann equation on the W^* algebra, a Markovian master equation is derived. It is shown that if the interaction potential is in the form of the JC interaction potential, then this Markovian master equation reduces to the Lindblad form. Finally, by a series of examples the ultra-high Q quantum cavity master equation is derived.

1. Introduction

The quantum optical cavity forms a very robust and rich system in which to study quantum effects at very easily measurable scales. Even with low Q cavities and strong bath interactions, it is possible to obtain nonclassical behavior in the cavity. With modern developments in cavity construction (examples of such are quantum dots (1) and Bose-Einstein condensates in lattices (2)), previously unattainable cavity levels are now open to experimentation. Direct applications of these new developments in cavity construction can be seen in the use of ultra-high Q micromasers, which started making an appearance in the early 90's (3). These micromasers use low density beams of Rydberg atoms injected in such a way such that only one atom is inside the cavity at any moment. This gives rise to the increased influence of quantum fluctuations inside the cavity. Since the resonant frequencies of Rydberg atoms are in the microwave region, an initial problem would be photon detection in the cavity. However it is possible to use the Rydberg atom beam as the probe pulse as well. By measuring the final states of the atoms at exit, it is possible to obtain indirect information about the cavity. In this setting, tests of Quantum Electrodynamics, and Quantum Measurement theory can be conducte. Because of the entanglement of the atom and field, it is also possible to conduct quantum information and computation experiments.

While the variety of applications of the single atom cavity resonator are numerous and varied, the models that describe the system are well defined and have analytic solutions. This can be demonstrated by the simplicity of the Jaynes-Cummings model (JCM) (4) (developed by Jaynes and Cummings to show Rabbi flops in different systems), which accurately describes (to a first order perturbative limit) the modern micromaser, but is just a twostate Hamiltonian with a Markovian interaction potential. Because the JCM is Markovian and

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limited to a twostate atom-field, it is possible to obtain analytic solutions to the JCM. It is of note that these solutions show a completely reversible Rabbi state flopping (see (5) for an analysis of the Rabbi flops and reversibility). Couple this fact with the nature of the perfect or ultra-high Q cavity and one can see the attractiveness of the model. We derive the JC interaction potential using quantized electromagnetic modes inside the cavity and the corresponding interaction with a strong dipole rigid rotor. The JCM directly follows from this derivation, providing both the multi-photon and twostate models.

The JCM works very well in the perfect or ultra-high Q cavities, but it falls short when the system needs to interact with the outside world. This is because it becomes difficult to create cavities with a low enough coupling to the laboratory heat bath. Recently this low level of coupling has become accessible in the form of low temperature cavities. It is possible to use a perturbative approach to the problem (see (6)). However, there is a point when the cavity interacts strongly enough with the out side world that perturbative analysis fails. At this point it is necessary to either use a Quantum Stochastic Langevin equation or the Liouville-von Neumann (as seen in section 6) master equation. There are advantages in using either method. Some of these advantages are discussed in the following sections.

Quantum Stochastics was offered as an alternate to the usual commutator derivation for the dynamics of Quantum cavities. Using a stationary quantum white noise as the heat bath input, and a first order Markov approximation, it is possible to use the Heisenberg equations of motion for an arbitrary system operator. Also, if an appropriate Wiener process is chosen, one can back derive the Ito calculus rules using the Heisenberg equation previously derived. It can be shown that it is possible to couple the Ito calculus with the trace representation of the expectation value to derive the high Q master equation. However we only pursue the stochastic model in a brief investigation of the Heisenberg equations.

The theory of Completely Positive Quantum Semigroups holds many insights into quantum information theory. Early work by Lindblad goes into entropy inequalities, and moves into Markovian dynamics through generating operators on a completely dissipative * map (which we take to be a W^* algebra). Through the theory of completely positive semigroups, with the relations of completely dissipative maps, an arbitrary Markovian master equation can be defined. It has been further shown that this Markovian master equation is even well defined on a C^* algebra. From the interaction picture, and the Liouville-von Neumann equation, Markovian and non-Markovian master equations can be derived. The method relies on a W^* algebra and the class of trace Banach operators to provide effective models. It is shown that under the assumption that the interaction potential (between the bath and system) is of a similar form as the JC interaction model, then it is possible to see that an arbitrary Markovian master equation through the Liouville-von Neumann equation is equivalent to the Lindblad form. Finally, it is shown that with an appropriate interaction potential, the Liouville-von Neumann equation reduces to the quantum optical cavity master equation.

2. Formalism

For convenience, most of the mathematical definitions will be made here. To start, let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ the W^* algebra of all bounded operators on $\mathcal{H}$. Also let $T(\mathcal{H})$ be the class of trace operators, such that $T(\mathcal{H})$ is a Banach space with trace norm:

$$\|\rho\|_1 = Tr(\rho),$$

where ρ is a member of the class of density matrices on $\mathcal{H}$ such that $\rho \in T(\mathcal{H})$. On $T(\mathcal{H})$, the trace is defined as:

Definition 1. Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ be a bounded W^* operator and let there exist an orthonormal basis $\{|1\rangle, |2\rangle, \dots, |n\rangle, \dots\} \in \mathcal{H}$. The trace operator is defined such that $Tr \in T(\mathcal{H})$ as $Tr(\mathcal{A}) = \sum_i \langle i|\mathcal{A}|i\rangle$ with the relations

(2.1a) $Tr(\alpha\mathcal{A}_1 + \beta\mathcal{A}_2) = \alpha Tr(\mathcal{A}_1) + \beta Tr(\mathcal{A}_2)$,

(2.1b) $Tr(\mathcal{A}_1\mathcal{A}_2) = Tr(\mathcal{A}_2\mathcal{A}_1)$,

(2.1c) $Tr(\mathcal{A}) = \sum eig(\mathcal{A})$,

(2.1d) $Tr(\mathcal{A}^*) = \overline{Tr(\mathcal{A})}$.

Then, the class of density matrices are defined as:

Definition 2. Let $H \in \mathcal{B}(\mathcal{H})$ be the interaction Hamiltonian for an arbitrary system and let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ be a bounded W^* operator, then the density matrix $\rho \in T(\mathcal{H})$ is defined as

(2.2a) $\rho = \sum_j p_j |j\rangle\langle j|$, p_j is the probability of state j,

(2.2b) $p_j > 0, \ \sum_j p_j = 1 \Rightarrow \rho \in CP(\mathcal{H}), \ \|\rho\|_1 = 1$,

(2.2c) $Tr(\rho\mathcal{A}) = \langle\mathcal{A}\rangle \ \forall \mathcal{A} \in \mathcal{B}(\mathcal{H})$,

(2.2d) and ρ is given by the Liouville-von Neumann equation: $\dot{\rho} = \dfrac{1}{i\hbar}[H, \rho]$.

Definition 3. Assume a boson Fock space $\Gamma(\mathcal{H})$ such that $|n\rangle$ are boson states for a given $\omega \in \mathbb{R}$ representing the frequency of the photon space. Then it follows that the boson creation and annihilation operators $\varrho^\dagger, \varrho \in \mathcal{B}(\mathcal{H})$ and $\varrho^\dagger, \varrho : \omega \to \mathcal{B}(\mathcal{H})$ operate on the Fock space as

(2.3a) $\varrho(\omega')|n\rangle = \partial_-^n \delta(\omega - \omega')|n-1\rangle$

(2.3b) $\varrho^\dagger(\omega')|n\rangle = \partial_+^n \delta(\omega - \omega')|n+1\rangle$

(2.3c) $\varrho^\dagger(\omega')\varrho(\omega')|n\rangle = \partial_-^n \partial_+^{n-1} \delta(\omega - \omega')|n\rangle = n\delta(\omega - \omega')|n\rangle$

(2.3d) $\varrho(\omega')\varrho^\dagger(\omega')|n\rangle = \partial_-^{n+1} \partial_+^{n} \delta(\omega - \omega')|n\rangle = (n+1)\delta(\omega - \omega')|n\rangle$

where $n = 1/(\exp(\hbar\omega/kT) - 1)$, and $\varrho^\dagger, \varrho$ follow the commutator algebra:

(2.4a) $[\varrho(\omega), \varrho^\dagger(\omega')] = \delta(\omega - \omega')$

(2.4b) $[\varrho(\omega), \varrho(\omega')] = [\varrho^\dagger(\omega), \varrho^\dagger(\omega')] = 0$.

3. The Jaynes-Cummings Model

Here we derive the simple Jaynes-Cummings interaction Hamiltonian (4) from the perspective of a quantum rigid rotor. This approach uses simple techniques to derive the standard twostate Hamiltonian from first principles without a priori knowledge of its form.

Starting with a quantum rigid rotor, with a dipole term $\mathcal{D} \in \mathbb{R}^3$ in a cavity with electric field modes $\mathcal{E} \in \mathbb{R}^3$ where $\mathcal{E} : \omega \to \mathbb{R}^3$; the Hamiltonian of the rigid rotor and cavity with the electromagnetic interaction term can be written down as

$$H = -\frac{\hbar^2}{2m} L^2 + \hbar \int d\omega \ \{\hbar\Omega(\omega) a^\dagger a + \langle \mathcal{D}, \mathcal{E} \rangle(\omega)\}. \tag{3.1}$$

Where $a^\dagger, a$ are the boson creation and annihilation operators given by Definition 3. We can expand the field modes in terms of the creation and annihilation operators as $\mathcal{E} =$

$\mathcal{E}_0(a^\dagger + a), \mathcal{E}_0 \in \mathbb{R}^3$ and $\mathcal{E}_0 : \omega \to \mathbb{R}^3$ which gives

$$H = -\frac{\hbar^2}{2m}L^2 + \hbar \int d\omega \ \hbar\Omega(\omega)a^\dagger a + V.$$

Where V is the interaction potential

$$V(\omega) = \hbar \int d\omega \ \langle \mathcal{D}, \mathcal{E}_0 \rangle (a^\dagger + a) \tag{3.2}$$

Theorem 1. *With a basis of $|lmn\rangle = |lm\rangle \otimes |n\rangle$, where $|lm\rangle, |n\rangle \in \mathcal{H}$ are the spherical harmonics and the Fock space $\Gamma(\mathcal{H})$ respectively, equation 3.2 becomes*

$$V(\omega) = \hbar \int d\omega \ \|\mathcal{D}\| \|\mathcal{E}_0\| (a^\dagger + a)(B_l^- + B_l^-). \tag{3.3a}$$

Then under the rotating wave approximation the multi-photon Jaynes Cummings interaction potential becomes

$$V = \hbar \sum_i g_i (a_i^\dagger B_l^- + a_i B_l^+), \tag{3.3b}$$

where $B_l^\pm$ are the orbital angular momentum ladder operators

Proof. Expand the inner product in $V(\omega)$ (note that $\cos(\theta) = P_1^0 \propto |1,0\rangle$) and recall the Wigner-Eckhart Theorem

$$\begin{aligned}\langle l'm'n'|V(\omega)|lmn\rangle &= \langle l'm'n'|\hbar \int d\omega \ \{\|\mathcal{D}\| \|\mathcal{E}(\omega)\|\} P_1^0 |lmn\rangle \\ &= \langle l'm'|10, jm\rangle \langle l'n'|\hbar \int d\omega \ \{\|\mathcal{D}\| \|\mathcal{E}(\omega)\|\} |ln\rangle.\end{aligned}$$

Thus by the triangle Theorem for angular momentum we obtain the transition rules for the interaction potential to be $|j' - j| = 1, 0$. Ignoring any zero transition contributions, we can rewrite the interaction potential in terms of the orbital momentum ladder operators $B_l^\pm \in \mathcal{B}(\mathcal{H}) \cap \mathcal{A}_l$, where $\mathcal{A}_l$ is a nonlinear Lie algebra (7; 8; 9; 10). Substituting the ladder operators for the P_1^0 term, and gathering the constants into a single coupling constant, we get

$$V(\omega) = \hbar \int d\omega \ \|\mathcal{D}\| \|\mathcal{E}_0\| (a^\dagger + a)(B_l^- + B_l^-).$$

If we apply the rotating wave approximation (seen in (11)) then:

$$(a^\dagger + a)(B_l^- + B_l^-) \to (a^\dagger B_l^- + a B_l^+).$$

Let $2\|\mathcal{D}\| \|\mathcal{E}_0\| = g(\omega)$ and note transitions of the rigid rotor form quantized energy states $(\delta E_j = j(j+1))$. Thus interaction potential becomes

$$V = \hbar \sum_i g_i (a_i^\dagger B_l^- + a_i B_l^+).$$

□

Carrying the quantized energy states in the cavity through the rest of the Hamiltonian, we arrive at the multi-photon Jaynes-Cummings model

$$H = -\frac{\hbar^2}{2m}L^2 + \sum_i \{\hbar\Omega_i a_i^\dagger a_i + \hbar g_i (a_i^\dagger B_l^- + a_i B_l^+)\}. \tag{3.4}$$

It is a simple matter to extend the rotating wave approximation through the rest of the Hamiltonian, by replacing $-\hbar^2 L^2/2m$ with $\hbar\omega\sigma_z/2$. Where $\sigma_z/2$ is the Pauli z spin matrix and the energy of the transition is ω. Also simplify by replacing B_l^-, B_l^+ with σ_- and σ_+

respectively[1], resulting with

$$H = \hbar\omega\sigma_z + \hbar\Omega a^\dagger a + \hbar g(a^\dagger\sigma_- + a\sigma_+), \tag{3.5}$$

which is the classic twostate Jaynes-Cummings model.

Remark 1. Equation 3.5 was derived using a simple rigid rotor with a fixed dipole moment. However, the result is a completely general equation that is applicable to any twostate system that has either a permanent or induced dipole moment. It is also useful to note that the boson operators are not constrained to being members of the cavity operator space. It will be shown that this kind of interaction holds for non-cavity operators as well.

The analysis of equation 3.5, while simple, is not the focus of this paper. It is notable that the eigenfunctions of equation 3.5 show Rabbi flopping, and predict reversible states. In modern micromaser experiments, this is indeed the case. Often it is only necessary to do a perturbative analysis of the expectation values to achieve accurate results for very high Q systems. For a detailed analysis of the Jaynes-Cummings solutions, see Dutra (11) and Meystre (12). The solutions are also discussed in a more applied context in Shore and Knight (5) and Meystre and Sargent (3).

4. Quantum Stochastic Langevin Equation for Cavity-Heat Bath systems

The use of a Langevin equation is one of the simplest ways to include bath-system dynamics. It has the benefit of easily showing dynamical characteristics of the system, and lends itself to linear systems easily. The Langevin approach has the disadvantage of most Heisenberg systems, in that one can only examine the evolution of operators, and not the states directly.

Assume that the Hamiltonian $H \in \mathcal{B}(\mathcal{H})$ can be written as $H = H_{sys} + H_b + H_{int}$; $H_{sys}, H_b, H_{int} \in \mathcal{B}(\mathcal{H})$ where the system Hamiltonian is left undefined, and the bath & interaction Hamiltonians are defined as

$$H_b = \hbar \int_{-\infty}^{\infty} d\omega' \; \omega' b^\dagger(\omega') b(\omega') \tag{4.1}$$

$$H_{int} = i\hbar \int_{-\infty}^{\infty} d\omega' \; \kappa(\omega')(b^\dagger(\omega')c - c^\dagger b(\omega')). \tag{4.2}$$

Here c is a bounded system operator such that $c \in \mathcal{B}(\mathcal{H})$, $b^\dagger(\omega), b(\omega)$ are the boson creation and annihilation operators given in Definition 3 and $\kappa(\omega)$ is the cavity coupling constant. With the addition that we also take $[b(\omega), c] = [b^\dagger(\omega'), c] = 0$.

Then the Heisenberg equations of motion for $b(\omega)$ and a (another arbitrary system operator, $a \in \mathcal{B}(\mathcal{H})$) are

$$\dot{b}(\omega) = -\frac{i}{\hbar}[b(\omega), H] = -\frac{i}{\hbar}[b(\omega), H_b + H_{int}] \tag{4.3}$$

$$\dot{a} = -\frac{i}{\hbar}[a, H] = -\frac{i}{\hbar}[a, H_{sys} + H_{int}]. \tag{4.4}$$

Before these can be solved, several commutators must be worked out.

Theorem 2. *Assuming Definition 3 applies, then the following equation holds*

$$\frac{1}{\hbar}[b(\omega), H_b] = \omega b(\omega). \tag{4.5}$$

[1] $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the non-hermitian spin flip matrices

Proof. It can be seen, that

$$\frac{1}{\hbar}[b(\omega), H_b] = \int_{-\infty}^{\infty} d\omega'\ \omega'\{b(\omega)b^\dagger(\omega')b(\omega') - b^\dagger(\omega')b(\omega')b(\omega)\}. \tag{4.6}$$

Then operating on equation 4.6 with $\langle n'|$ & $|n\rangle$ we get

$$\langle n'| \int_{-\infty}^{\infty} d\omega'\ \omega'\{b(\omega)b^\dagger(\omega')b(\omega') - b^\dagger(\omega')b(\omega')b(\omega)\}|n\rangle =$$

$$\langle n'| \int_{-\infty}^{\infty} d\omega'\ \omega'\{\partial_-^n n - (n-1)\partial_-^n\}\delta(\omega-\omega')|n-1\rangle = \langle n'|\omega\partial_-^n|n-1\rangle = \langle n'|\omega b(\omega)|n\rangle.$$

□

Theorem 3. *Assume Definition 3, then the equation*

$$\frac{1}{i\hbar}[b(\omega), H] = \kappa(\omega)c \tag{4.7}$$

holds.

Proof. As before, operating on equation 4.7 with $\langle n'|$ & $|n\rangle$ and taking $b(\omega)$ through the integral we get

$$\frac{1}{i\hbar}\langle n'|i\hbar \int_{-\infty}^{\infty} d\omega'\ \kappa(\omega')\{b(\omega)(b^\dagger(\omega')c - c^\dagger b(\omega')) - (b^\dagger(\omega')c - c^\dagger b(\omega'))b(\omega)\}|n\rangle =$$

$$\int_{-\infty}^{\infty} d\omega'\ \kappa(\omega')\{\langle n'|\partial_-^{n+1}\partial_+^n c|n\rangle - \langle n'|c^\dagger\partial_-^n\partial_-^{n-1}|n-2\rangle -$$
$$\langle n'|\partial_+^{n-1}\partial_-^n c|n\rangle + \langle n'|c^\dagger\partial_-^n\partial_-^{n-1}|n-2\rangle\}\delta(\omega-\omega') =$$
$$\kappa(\omega)\{\langle n'|(n+1)c|n\rangle - \langle n'|nc|n\rangle\} = \kappa(\omega)c.$$

□

Using Theorems 2 and 3 we can easily solve the Heisenberg equation for $b(\omega)$

$$\dot{b} = -i\omega b(\omega) + \kappa(\omega)c \Rightarrow b(\omega) = e^{-i\omega(t-t_0)}b_0(\omega) + \int_{t_0}^{t} dt'\ e^{-i\omega(t-t')}\kappa(\omega)c(t'). \tag{4.8}$$

If we make the first Markov approximation (13)

$$\kappa(\omega) = \sqrt{\gamma/2\pi}$$

then equation 4.4 becomes

$$\begin{aligned}\dot{a} &= -\frac{i}{\hbar}[a(t), H_{sys}] + \kappa \int_{-\infty}^{\infty} d\omega'\ ([a(t'), b^\dagger(\omega')c(t')] - [a(t'), c^\dagger(t')b(\omega')])\\ &= -\frac{i}{\hbar}[a(t), H_{sys}] + \kappa \int_{-\infty}^{\infty} d\omega'\ (b^\dagger(\omega')[a(t'), c(t')] - [a(t'), c^\dagger(t')]b(\omega')).\end{aligned}$$

Substituting in equation 4.8, we get

(4.9)
$$\dot{a} = -\frac{i}{\hbar}[a(t), H_{sys}] + \int_{-\infty}^{\infty} d\omega'\ \{\kappa(e^{i\omega(t-t_0)}b_0^\dagger(\omega)[a(t'), c(t')] - [a(t'), c^\dagger(t')]e^{-i\omega(t-t_0)}b_0(\omega))$$
$$+ \int_{t_0}^{t} dt'\ \kappa^2(e^{i\omega(t-t')}c^\dagger(t')[a(t'), c(t')]) + [a(t'), c^\dagger(t')]e^{-i\omega(t-t')}c(t'))\}.$$

Define a bath input field

$$b_{in}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \; e^{-i\omega(t-t_0)} b_0^\dagger(\omega) \tag{4.10}$$

and make use of the identities

$$\int_{-\infty}^{\infty} d\omega' \; e^{-i\omega(t-t_0)} = 2\pi\delta(t-t'), \text{ and by } c \in \mathcal{B}(\mathcal{H}) \Rightarrow \int_{t_0}^{t} dt' \; c(t')\delta(t-t') = c(t)/2$$

so that equation 4.9 becomes

$$\dot{a} = -\frac{i}{\hbar}[a(t), H_{sys}] + \sqrt{\gamma}(b_{in}^\dagger(t)[a(t), c(t)] - [a(t), c(t)]b_{in}(t)) + \frac{\gamma}{2}\{c^\dagger(t)[a(t), c(t)] - [a(t), c^\dagger(t)]c(t)\}$$

or, collecting terms

$$\dot{a} = -\frac{i}{\hbar}[a(t), H_{sys}] - \{[a(t), c^\dagger(t)](\frac{\gamma}{2}c(t) + \sqrt{\gamma}b_{in}(t)) - (\frac{\gamma}{2}c^\dagger(t) + \sqrt{\gamma}b_{in}^\dagger(t))[a(t), c(t)]\}. \tag{4.11}$$

Equation 4.11 is a common form for the Quantum Langevin equation for a high Q, low bath coupling system. The derivation shown is a detailed account of the one seen in Gardiner and Collet (13). the analysis of equation 4.11 can also be found in Gardiner and Collet (13).

5. Quantum Dynamical Semigroups

It is possible to completely define a Quantum Markovian master equation using the theory of completely dissipative generators on a CP Quantum Semigroup. This is done in the following discussion by using the mathematical construct developed by Lindblad (14), then completed by showing that the equation 6.8a satisfies the Lindblad form.

A Quantum Semigroup is defined as

Definition 4 (Ingarden & Kossakowski semigroup axioms (14))**.** Let $\mathcal{A}$ be a bounded W^*-algebra. A dynamic semigroup is a one parameter Hamiltonian Φ_t so that $\Phi_t : \mathcal{A} \to \mathcal{A}$ while being norm continuous. Also there is a bounded map $L : \mathcal{A} \to \mathcal{A}$ such that:

$\Phi_t \in CP(\mathcal{A})$,	$\Phi_t(I) = I$,
$\Phi_s \otimes \Phi_t = \Phi_{s+t}$,	$\lim_{t\to 0} \lVert\Phi_t - I\rVert = 0$,
Φ_t is ultra-weakly continuous wrt $\Phi_t : \mathcal{A} \to \mathcal{A}$,	$\Phi_t = e^{tL}$,
$\lim_{t\to 0} \lVert L - (\Phi_t - I)/t\rVert = 0$,	L is ultra-weakly continuous wrt $L : \mathcal{A} \to \mathcal{A}$.

We can further specify the nature of the Quantum Semigroup by calling the corollary

Theorem 4 (Lindblad (14) Corollary 1)**.** *For $\mathcal{A} \in \mathcal{B}(\mathcal{H})$, L is a bounded *-map $L : \mathcal{A} \to \mathcal{A}$, $\Phi_t = e^{tL}$ and $\Phi_t \in CP(\mathcal{A})$ iff $L \in CD(\mathcal{A})$. Then Φ_t is a norm continuous dynamical semigroup on $\mathcal{A}$ such that $\Phi_t : \mathcal{A} \to \mathcal{A}$ iff $\Phi_t = e^{tL}$.*

With the Quantum Semigroup thus defined, then the form of a completely dissipative generator in a CP Quantum semigroup follows from the proposition

Corollary 1 (Lindblad (14) Proposition 5-6)**.** *If $\mathcal{A}$ is a hyper finite factor of the separable Hilbert space $\mathcal{H}$, and $L \in CD(\mathcal{A})$, then there is a $\Psi \in CP(\mathcal{A})$ and a s.a $\mathcal{V} \in \mathcal{A}$ such that for all $X \in \mathcal{A}$*

$$L(X) = \Psi(X) = \frac{1}{2}\{\Psi(I), X\} + i[\mathcal{V}, X]. \tag{5.1}$$

By the proceeding Corollary, the Shrödinger picture Markovian Generator (or master equation) is

Corollary 2 (Lindblad (14) Theorem 2.). *$L \in CD(\mathcal{H})$ iff it is of the form*

$$L(\rho) = -i[\mathcal{V}, \rho] + \frac{1}{2}\sum([V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]), \tag{5.2}$$

which is the Lindblad form for the master equation, where $V_j, \sum V_j^\dagger V_j \in \mathcal{B}(\mathcal{H})$, $\mathcal{V}$ s.a. $\in \mathcal{B}(\mathcal{H})$.

It is of interest that this form of the master equation was first proposed for an N level atom by Davies (15) and independently of Lindblad by Gorini *et al* (16) for a finite-dimensional Hilbert space.

Remark 2. Following the aside made by Lindblad (14): Let $\mathcal{A}$ be a hyper finite factor of the separable Hilbert space $\mathcal{H}$. If $L : \mathcal{A} \to \mathcal{A}$ is a bounded ultra-weakly continuous *-map such that for all $X \in \mathcal{A}$ there is a positive normal map $\Psi : \mathcal{A} \to \mathcal{A}$ and $K \in \mathcal{A}$ such that $L(X) = \Psi(X) + KX + XK^\dagger$. Then the most general form of the Markovian Generator is:

Theorem 5 (Lindblad (14) Theorem 3.). *Let $\mathcal{A}$ be a C^*-algebra, $L : \mathcal{A} \to \mathcal{A}$ a bounded *-map and put $\Phi_t = e^{tL}$. If $L(X) = \Psi(X) + KX + XK^\dagger$ where $K \in \mathcal{A}$ and $\Psi \in CP(\mathcal{A})$, then $\Phi_t \in CP(\mathcal{A})$.*

This Theorem is of interest mathematically, but holds very little bearing on physical systems. Because of this, we will instead use Corollary 2 to derive the master equation as seen in the following section.

6. Liouville-von Neumann Markovian Master Equation

Using the interaction picture with a self adjoint Hamiltonian $H \in \mathcal{B}(\mathcal{H})$ which can be represented as $H = H_{sys} + \mathcal{V}$; $H_{sys}, \mathcal{V} \in \mathcal{B}(\mathcal{H})$, where H_{sys} is the collection of appropriate system operators, and $\mathcal{V} : t \to \mathcal{H}$ is the bath-system interaction potential, the Liouville-von Neumann equation (17) gives the dynamics of the density matrix by

$$\frac{\partial \kappa(t)}{\partial t} = \frac{1}{i\hbar}[\mathcal{V}, \kappa(t)]. \tag{6.1}$$

Where $\kappa \in T(\mathcal{H})$ satisfies Definition 2 and is the time dependent system density matrix.

We integrate formally to get

$$\kappa(t) - \kappa(t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt'[\mathcal{V}(t'), \kappa(t')],$$

then iterate using the von Neumann series with a kernel of $K\{\kappa\} = [\mathcal{V}, \kappa]$

$$\begin{aligned}
\kappa_1(t) &= \kappa(t_0) \\
\kappa_2(t) &= \frac{1}{i\hbar}\int_{t_0}^{t} dt'[\mathcal{V}(t'), \kappa(t_0)] \\
\kappa_3(t) &= -\frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''[\mathcal{V}(t'), [\mathcal{V}(t''), \kappa(t'')]]
\end{aligned}$$

to arrive at the the closed form solution

$$\kappa(t) - \kappa(t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt'[\mathcal{V}(t'), \kappa(t_0)] - \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''[\mathcal{V}(t'), [\mathcal{V}(t''), \kappa(t'')]]. \tag{6.2}$$

Separate the system density matrix into a combined bath and system matrix in the form: $\kappa(t) = \rho_s(t) \otimes \rho_b(t)$, with $\rho_s \in T(\mathcal{H}_s)$, $\rho_b \in T(\mathcal{H}_b)$ being the system and bath density matrices respectively[2], and that ρ_s and ρ_b satisfy Definition 2. The dynamical equation for $\kappa(t)$ becomes

$$\rho_s(t) \otimes \rho_b(t) - \rho_s(t_0) \otimes \rho_b(t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt'[\mathcal{V}(t'), \rho_s(t_0) \otimes \rho_b(t_0)] - \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''[\mathcal{V}(t'), [\mathcal{V}(t''), \rho_s(t'') \otimes \rho_b(t'')]].$$

Taking the trace over the bath operators

$$\rho_s(t) - \rho_s(t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt' Tr_b[\mathcal{V}(t'), \rho_s(t_0) \otimes \rho_b(t_0)] - \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' Tr_b[\mathcal{V}(t'), [\mathcal{V}(t''), \rho_s(t'') \otimes \rho_b(t'')]]$$

where we note that $Tr_b(\rho_s \otimes \rho_b) = \rho_s Tr_b(\rho_b) = \rho_s$.

Assume that we can write $\mathcal{V}$ in the general form $\mathcal{V} = \hbar \sum_i Q_i F_i$ where Q_i, F_i are the system and bath interaction operators respectively. Thus

$$\rho_s(t) - \rho_s(t_0) = -\int_{t_0}^{t} dt' Tr_b \sum_i [Q_i(t')F_i(t'), \rho_s(t_0) \otimes \rho_b(t_0)] - \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' Tr_b \sum_{\substack{i,j \\ j\neq i}} [Q_i(t')F_i(t'), [Q_j(t'')F_j(t''), \rho_s(t'') \otimes \rho_b(t'')]]. \tag{6.3}$$

Theorem 6. *Under the assumption of a stationary quantum white noise in the bath (see Gautam and Gardiner (17; 13) for further application, and the discussion in Gardiner, Huang and Attal (13; 18; 19)), then under the relations*

$$\langle F_i(t) \rangle = 0, \ \frac{d\rho_b(t)}{dt} = 0 \ \Rightarrow \ \rho_b(t) = \rho_b(t_0) \forall t$$

the first integral in equation 6.3 is

$$\int_{t_0}^{t} dt' Tr_b \sum_i [Q_i(t')F_i(t'), \rho_s(t_0) \otimes \rho_b(t_0)] = 0. \tag{6.4}$$

Proof. Expand equation 6.4 by writing out the commutator and taking the trace through the system operators to get

(6.5)

$$\begin{aligned}\rho_s(t) - \rho_s(t_0) &= -\sum_i \int_{t_0}^{t} dt' \{Q_i(t') Tr_b(F_i(t')\rho_b(t_0))\rho_s(t_0) - \rho_s(t_0) Q_i(t') Tr_b(\rho_b(t_0)F_i(t')\} \\ &= -\sum_i \int_{t_0}^{t} dt' \{Q_i(t')\langle F_i(t_0)\rangle \rho_s(t_0) - \rho_s(t_0) Q_i(t') \langle F_i(t_0)\rangle\}] \\ &= -\sum_i \int_{t_0}^{t} dt' [Q_i(t'), \rho_s(t_0)]\langle F_i(t_0)\rangle.\end{aligned}$$

By the assumptions made in the Theorem, equation 6.5 must equal zero. □

[2]Here you can add in a system-bath correlation term, however in the appendix of Gautam (17) it is shown that the correlation term is of higher order than what we have taken von Neumann series to.

Thus our dynamical equation for $\rho_s(t)$ becomes

$$\rho_s(t) - \rho_s(t_0) = -\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' Tr_b \sum_{\substack{i,j \\ j\neq i}} [Q_i(t')F_i(t'), [Q_j(t'')F_j(t''), \rho_s(t'') \otimes \rho_b(t_0)]].$$

Going from an integral equation to a differential equation for $\rho_s(t)$, expand the commutators and apply equation 2.1b:

$$\dot{\rho}_s = -\int_{t_0}^{t} dt' \sum_{\substack{i,j \\ j\neq i}} \{(Q_i(t)Q_j(t')\rho_s(t') - Q_j(t)\rho_s(t')Q_i(t'))\langle F_i(t)F_j(t')\rangle$$
$$- (Q_i(t')\rho_s(t')Q_j(t) - \rho_s(t')Q_j(t')Q_i(t))\langle F_j(t')F_i(t)\rangle\}.$$

Up to this point, we have followed the method of Gautam (17). However, instead of proceeding to obtain the non-Markovian master equation, we will make the Markov approximation to recover the master equation shown by Meystre *et al*(3; 12) Gardiner (13) and used by Savage and Carmichael's groups (20; 21; 22), which will then be generalized to show that this is equivalent to the Lindblad form (14; 15; 16) of the quantum master equation on a quantum dynamical semigroup.

Directly write the time dependence of $Q_n(\tau)$ in the interaction picture as

$$Q_n(\tau) = U(\tau)Q_nU^\dagger(\tau) = e^{iH_{sys}\tau/\hbar}Q_n e^{-iH_{sys}\tau/\hbar} = Q_n e^{i\omega_n\tau} \tag{6.6}$$

and apply the Markov approximation $\rho_s(t') = \rho_s(t)\ \forall t$ to get the master equation

$$\dot{\rho}_s = -\int_{t_0}^{t} dt' \sum_{\substack{i,j \\ j\neq i}} \{(Q_iQ_j\rho_s(t) - Q_j\rho_s(t)Q_i)e^{i(\omega_i t+\omega_j t')}\langle F_i(t)F_j(t')\rangle$$
$$- (Q_i\rho_s(t)Q_j - \rho_s(t)Q_jQ_i)e^{i(\omega_i t+\omega_j t')}\langle F_j(t')F_i(t)\rangle\}$$

or

$$\dot{\rho}_s = -\sum_{\substack{i,j \\ j\neq i}} \{(Q_iQ_j\rho_s(t) - Q_j\rho_s(t)Q_i)\Phi^r_{ij} - (Q_i\rho_s(t)Q_j - \rho_s(t)Q_jQ_i)\Phi^l_{ji}\} \tag{6.7}$$

where we have grouped the time dependent terms into the bath fields

$$\Phi^r_{ij} = \int_0^t d\tau e^{i(\omega_i t+\omega_j t')}\langle F_i(t)F_j(t')\rangle,\ \Phi^l_{ij} = \int_0^t d\tau e^{i(\omega_i t+\omega_j t')}\langle F_j(t')F_i(t)\rangle.$$

This can be further simplified if we make the change of variables $t' \to t - \tau$ and take the secular approximation, which is reasonable in the weak system-bath coupling limit. The master equation, equation 6.7, in the Shrödinger picture and the bath fields become:

$$\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \sum_{\substack{i,j \\ j\neq i^j}} \{(Q_iQ_j\rho_s(t) - Q_j\rho_s(t)Q_i)\Phi^r_{ij} - (Q_i\rho_s(t)Q_j - \rho_s(t)Q_jQ_i)\Phi^l_{ji}\} \tag{6.8a}$$

and

$$\Phi^r_{ij} = \int_{t_0}^t d\tau e^{-i\omega_j\tau}\langle F_i(t)F_j(t-\tau)\rangle,\ \Phi^l_{ij} = \int_{t_0}^t d\tau e^{-i\omega_j\tau}\langle F_j(t-\tau)F_i(t)\rangle. \tag{6.8b}$$

Before it is possible to expand this formulation to a CP Quantum Semigroup, it is necessary to further reduce the equations 6.8b. To do this, we write the bath operators in the Shrödinger picture, with the time dependence explicitly written as in equation 6.6:

$$F_n(\tau) = U(\tau)F_nU^\dagger(\tau) = F_n e^{i\Omega_n\tau}.$$

Then equations 6.8b become

$$\Phi^r_{ij} = \int_{t_0}^{t} d\tau e^{-i\omega_j\tau}\langle F_i e^{i\Omega_i t} F_j e^{-i\Omega_j(t-\tau)}\rangle = \langle F_i F_j\rangle \int_{t_0}^{t} d\tau e^{i(\Omega_i+\Omega_j)t} e^{-i(\omega_j+\Omega_j)\tau}$$

$$\Phi^l_{ij} = \int_{t_0}^{t} d\tau e^{-i\omega_j\tau}\langle F_j e^{-i\Omega_j(t-\tau)} F_i e^{i\Omega_i t}\rangle = \langle F_j F_i\rangle \int_{t_0}^{t} d\tau e^{i(\Omega_i+\Omega_j)t} e^{-i(\omega_j+\Omega_j)\tau}.$$

Again, make the secular approximation, assumption that $t \gg t_0$ and that the bath is tuned to the cavity, such that

$$\Phi^r_{ij} = \langle F_i F_j\rangle,\ \Phi^l_{ij} = \langle F_j F_i\rangle. \tag{6.9}$$

Proposition 1. Assuming a potential of similar form to the JCM, such that the relations $Q_{2k+1} = Q^\dagger_{2k},\ F_{2k} = F^\dagger_{2k+1}$ and the commutator identities:

$$[Q_{2k}, Q_j] = [F_{2k}, F_j] = \delta_{2k+1,j},\ [Q_{2k+1}, Q_j] = [F_{2k+1}, F_j] = \delta_{2k,j}$$

hold. Then the interaction potential becomes

$$\mathcal{V} = \hbar \sum_{k=0} Q_k F^\dagger_k + Q^\dagger_k F_k. \tag{6.10}$$

Theorem 7. *By Proposition 1, equation 6.8a becomes*

$$\mathcal{L}\rho_s = \dot{\rho}_s = -i[H_{sys}, \rho_s(t)] + \frac{1}{2}\sum_{k=0}\{[V_k\rho_s(t), V^\dagger_k] + [V_k, \rho_s(t)V^\dagger_k]\} \tag{6.11}$$

Proof. Proposition 1 simplifies equations 6.8a and 6.8b to:

$$\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \sum_k\{(Q_kQ^\dagger_k\rho_s(t) - Q^\dagger_k\rho_s(t)Q_k)\Phi^r_k - (Q_k\rho_s(t)Q^\dagger_k - \rho_s(t)Q^\dagger_kQ_k)\Phi^l_k +$$
$$(Q^\dagger_kQ_k\rho_s(t) - Q_k\rho_s(t)Q^\dagger_k)\Phi^l_k - (Q^\dagger_k\rho_s(t)Q_k - \rho_s(t)Q_kQ^\dagger_k)\Phi^r_k\}. \Rightarrow$$

$$-i[H_{sys}, \rho_s(t)] - \sum_k\{(Q^\dagger_kQ_k\rho_s(t) - 2Q_k\rho_s(t)Q^\dagger_k + \rho_s(t)Q^\dagger_kQ_k)\Phi^r_k + \tag{6.12a}$$
$$(Q_kQ^\dagger_k\rho_s(t) - 2Q^\dagger_k\rho_s(t)Q_k + \rho_s(t)Q_kQ^\dagger_k)\Phi^l_k\}.$$

$$\text{and } \Phi^r_k = \langle F_kF^\dagger_k\rangle,\ \Phi^l_k = \langle F^\dagger_kF_k\rangle \tag{6.12b}$$

Let $F_k = g_k\aleph_k$, where $g \in \mathbb{R},\ \aleph \in \mathcal{B}(\mathcal{H});\ (g, \aleph) : \omega \to \mathbb{R}$ and $\eta_k = 2g^2_k$ such that

$$\mathcal{V} = \hbar\sum_k(g_iQ^\dagger_k\aleph_k) + g^\dagger_kQ_k\aleph^\dagger_k, \tag{6.13}$$

and equation 6.12b becomes

$$\Phi^r_k = \frac{\eta}{2}\langle\aleph_k\aleph^\dagger_k\rangle,\ \Phi^l_k = \frac{\eta}{2}\langle\aleph^\dagger_k\aleph_k\rangle \tag{6.14}$$

Finally, let $V_{2k} = \sqrt{\eta\Phi^r_k}Q_k$ & $V_{2k+1} = \sqrt{\eta\Phi^l_k}Q^\dagger_k$ then $V^\dagger_{2k} = \sqrt{\eta\Phi^r_k}Q^\dagger_k$ & $V^\dagger_{2k+1} = \sqrt{\eta\Phi^l_k}Q_k$ so that the master equation can be described as:

$$\mathcal{L}\rho_s = \dot{\rho}_s = -i[H_{sys}, \rho_s(t)] + \frac{1}{2}\sum_{k=0}\{[V_k\rho_s(t), V^\dagger_k] + [V_k, \rho_s(t)V^\dagger_k]\}$$

□

Theorem 8. *Assuming that the system bath interaction potential can be written as*

$$\mathcal{V} = \hbar\sum_{k=0} Q_kF^\dagger_k + Q^\dagger_kF_k$$

where Q, F are defined as $Q, F \in \mathcal{B}(\mathcal{H})$ and Q, F follow Proposition 1. Then by the Definitions $\rho_s \in Tr(\mathcal{H})$ and $H_{sys} \in \mathcal{B}(\mathcal{H})$, equation 6.11 is a bounded * map and $\mathcal{L} \in CD(\mathcal{H})$.

Proof. $\mathcal{L}$ is a bounded * map by Definition ($\mathcal{L} \in \mathcal{B}(\mathcal{H})$), and is $CD(\mathcal{H})$ by Corollary 2. □

Corollary 3. *The equation*

$$\mathcal{L}\rho_s = \dot{\rho}_s = -i[H_{sys}, \rho_s(t)] + \frac{1}{2}\sum_{k=0}\{[V_k\rho_s(t), V_k^\dagger] + [V_k, \rho_s(t)V_k^\dagger]\} \tag{6.15}$$

is the general (but not necessarily unique, see (23)) Markovian master equation for a system such defined in Section 2 and at the beginning of Section 6.

Proof. By Theorems 4 & 8 and Corollary 2. □

Example 1. Let $Q = \omega a$, $\aleph = b$ where a & b are given by Definition 3 and $\kappa = \omega\eta/2$, then $\mathcal{V} = \hbar\omega g(ab^\dagger + a^\dagger b)$ and $V_0 = \sqrt{2\kappa(n+1)}a$, $V_1 = \sqrt{2\kappa n}a^\dagger$. Substitute this into equation 6.11 to get

$$\begin{aligned}\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \kappa\{n(aa^\dagger\rho_s(t) - 2a^\dagger\rho_s(t)a + \rho_s(t)aa^\dagger)+ \\ (n+1)(a^\dagger a\rho_s(t) - 2a\rho_s(t)a^\dagger + \rho_s(t)a^\dagger a)\}.\end{aligned} \tag{6.16}$$

Under the zero temperature limit, this reduces to

$$\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \kappa\{(a^\dagger a\rho_s(t) - 2a\rho_s(t)a^\dagger + \rho_s(t)a^\dagger a)\}. \tag{6.17}$$

Example 2. In Corollary to the previous example, the atom-bath interaction with an interaction potential (with $\eta \to \gamma$): $\mathcal{V} = \hbar\sqrt{\gamma}(\sigma_- b^\dagger + \sigma_+ b)$ such that $V_0 = \sqrt{\gamma(n+1)}\sigma_-$, $V_1 = \sqrt{\gamma n}\sigma_+$. Then equation 6.11 becomes

$$\begin{aligned}\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \frac{\gamma}{2}\{n(\sigma_-\sigma_+\rho_s(t) - 2\sigma_+\rho_s(t)\sigma_- + \rho_s(t)\sigma_-\sigma_+)+ \\ (n+1)(\sigma_+\sigma_-\rho_s(t) - 2\sigma_-\rho_s(t)\sigma_+ + \rho_s(t)\sigma_+\sigma_-)\}\end{aligned} \tag{6.18}$$

which becomes, in the zero temperature limit,

$$\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \frac{\gamma}{2}\{(\sigma_+\sigma_-\rho_s(t) - 2\sigma_-\rho_s(t)\sigma_+ + \rho_s(t)\sigma_+\sigma_-)\}. \tag{6.19}$$

Remark 3. Add equations 6.17 and 6.19 to obtain

$$\begin{aligned}\dot{\rho}_s = -i[H_{sys}, \rho_s(t)] - \frac{\gamma}{2}(\sigma_+\sigma_-\rho_s(t) - 2\sigma_-\rho_s(t)\sigma_+ + \rho_s(t)\sigma_+\sigma_-)- \\ \kappa(a^\dagger a\rho_s(t) - 2a\rho_s(t)a^\dagger + \rho_s(t)a^\dagger a)\end{aligned} \tag{6.20}$$

which is the cavity-bath photon density equation derived using Quantum Stochastics by Meystre (3) and used by Savage and Carmichael's groups (20; 22; 21) in their early experimental research.

7. Summary

The Jaynes-Cummings model for single and multiple photon systems has been developed for ultra-high Q quantum cavities. It was shown that the JC interaction potential is a member of $\mathcal{B}(\mathcal{H}) \cap \mathcal{A}_l$ and is of the form

$$\mathcal{V} = \hbar g(a^\dagger\sigma_- + a\sigma_+).$$

It was also shown that this potential can be generalized to the form

$$\mathcal{V} = \hbar\sum_k(g_i Q_k^\dagger\aleph_k + g_k^\dagger Q_k\aleph_k^\dagger),$$

where $g \in \mathbb{R}$ and $Q, \aleph \in B(\mathcal{H})$ and Proposition 1 applies.

A first order Markovian Langevin equation was derived from Heisenberg's equations, but it is seen that the the Heisenberg formulation is not ideal for this system. Using the theory of Completely Positive Quantum Semigroups, the general form for the Markovian master equation was presented. Through the use of the Liouville-von Neumann equation, a Markovian master equation was derived. Assuming the interaction potential to be equation 6.13, we showed that the Liouville-von Neumann master equation was also a Lindblad master equation. Thus it was demonstrated that the Jaynes-Cummings model and the Lindblad CD generators are linked by the Liouville-von Neumann equation on the W^* algebra.

Finally, by choosing the interaction potential to correspond to a cavity-bath system, the master equation used by Savage and Carmichael's groups (20; 22; 21) is recoverable as shown by examples 1 & 2 and remark 3. With equations 3.5 and 6.20, it is possible to accurately model the dynamics of quantum cavities. With micromaser applications, the JCM can be used to describe the dynamics while an atom is inside the cavity, and equation 6.17 is used for periods with no atom in the cavity.

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