# Instantons and Sphalerons in a Magnetic Field

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GB, G.Dunne & D. Kharzeev, arXiv:1112.0532, PRD **85**, 045026
GB & D. Kharzeev, arXiv:1202.2161, PRD **85**, 086012

# Outline

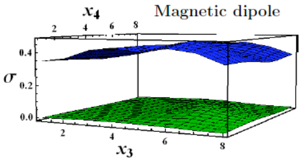
- Motivation & some lattice results
- General facts on Dirac operator
- Large instanton limit & dipole moments
- Sphaleron rate at strong coupling

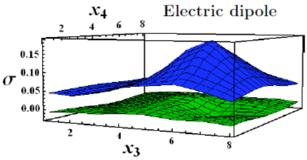
# Motivation & some lattice results

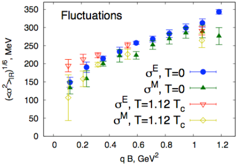
Interplay between topology & magnetic field

- Chiral magnetic effect $\vec{J} = q\frac{q\vec{B}}{2\pi}\frac{\mu_5}{\pi}$
- Anomaly induced hydrodynamics (chiral MHD), heavy ions etc...
- Lattice results
  - ITEP group (electric & dipole moments)
  - T. Blum et al. (zero modes $\propto$ B)
  - A. Yamamoto (C.M. conductivity)

(Polikarpov et al. '09)



# Notation & conventions

work in: $\mathbb{R}^4(\mathbb{T}^4)$

chiral basis: $\gamma_\mu = \begin{pmatrix} 0 & \alpha_\mu \\ \bar{\alpha}_\mu & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

$$\alpha_\mu = (\mathbb{1}, -i\vec{\sigma}) \quad , \quad \bar{\alpha}_\mu = (\mathbb{1}, i\vec{\sigma}) = \alpha_\mu^\dagger$$

Dirac operator: $\not{D} = \begin{pmatrix} 0 & \alpha_\mu \mathcal{D}_\mu \\ \bar{\alpha}_\mu \mathcal{D}_\mu & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & D \\ -D^\dagger & 0 \end{pmatrix}$

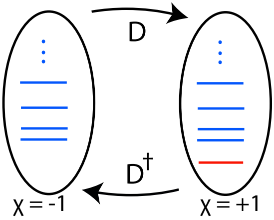
gauge field: $\mathcal{A}_\mu = A_\mu + a_\mu$

# Notation & conventions

diagonal form: $\left(i\not{D}\right)^2 \psi_\lambda = \begin{pmatrix} DD^\dagger & 0 \\ 0 & D^\dagger D \end{pmatrix} \psi_\lambda = \lambda^2 \psi_\lambda$

$$\begin{aligned} \chi = +1: \qquad & DD^\dagger = -\mathcal{D}_\mu^2 - \mathcal{F}_{\mu\nu}\bar{\sigma}_{\mu\nu} \\ \chi = -1: \qquad & D^\dagger D = -\mathcal{D}_\mu^2 - \mathcal{F}_{\mu\nu}\sigma_{\mu\nu} \end{aligned}$$

"supersymmetry:" for $\lambda \neq 0$, $DD^\dagger$ and $D^\dagger D$ has identical spectra



# Magnetic field background

$$DD^\dagger = D^\dagger D = -\mathcal{D}_\mu^2 - B\sigma_3 = -\partial_3^2 - \partial_4^2 - \mathcal{D}_\mp \mathcal{D}_\pm \pm B - B\,\sigma_3$$

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Zero modes: Definite spin, both chiralities

Number of zero modes: $N_+ = N_- = \frac{1}{2\pi}\int d^2x\, B$

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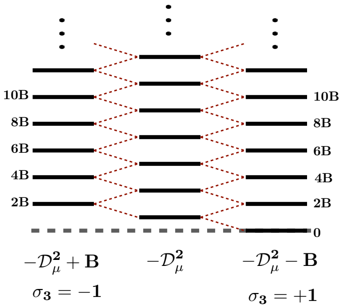
Constant field: $a_\mu = \frac{B}{2}(-x_2, x_1, 0, 0)$

Spectrum:



# Instanton (BPST) background

$$DD^\dagger = -\mathcal{D}_\mu^2 \qquad , \qquad D^\dagger D = -\mathcal{D}_\mu^2 - F_{\mu\nu}\sigma_{\mu\nu}$$

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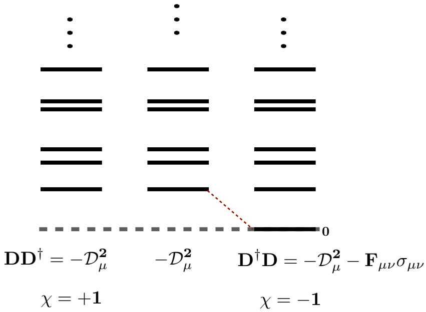

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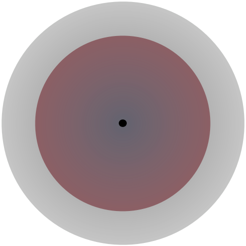
Competition between instanton and magnetic field

$\downarrow$ try to align chiralities (instanton)

$\downarrow$ align spins (magnetic field)

Instanton zero mode: $|\psi_0|^2 = \frac{64\,\rho^2}{(x^2+\rho^2)^3}$

Topological charge: $q_5(x) = \frac{192\,\rho^4}{(x^2+\rho^2)^4}$



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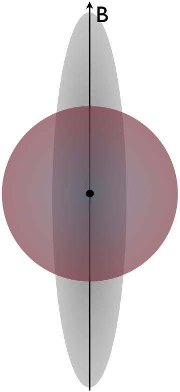

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$$A_\mu^a = 2\frac{\eta_{\mu\nu}^a\, x_\nu}{x^2+\rho^2} \approx \frac{2}{\rho^2}\eta_{\mu\nu}^a x_\nu + \ldots$$

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after appropriate gauge rotation & Lorentz transformation:

$$\mathcal{A}_\mu = -\frac{F}{2}(-x_2, x_1, -x_4, x_3)\tau^3 + \frac{B}{2}(-x_2, x_1, 0, 0)\mathbb{1}_{2\times 2}$$

quasi-abelian, covariantly constant $\rightarrow$ soluble!

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$$\mathcal{F}_{12} = \begin{pmatrix} B - F & 0 \\ 0 & B + F \end{pmatrix}$$

$$\mathcal{F}_{34} = \begin{pmatrix} -F & 0 \\ 0 & F \end{pmatrix}$$

Landau problem with field strengths $\mathcal{F}_{12}$ & $\mathcal{F}_{34}$

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Landau problem with field strengths $\mathcal{F}_{12}$ & $\mathcal{F}_{34}$

Topological charge: $\frac{1}{32\pi^2}\,\mathcal{F}^a_{\mu\nu}\tilde{\mathcal{F}}^a_{\mu\nu} = \frac{F^2}{2\pi^2}$

# Zero modes

$$\tau = -1$$

+ chirality: $DD^\dagger$

$$-\left(\mathcal{D}_1 - i\mathcal{D}_2\right)\left(\mathcal{D}_1 + i\mathcal{D}_2\right) - \left(\mathcal{D}_3 - i\mathcal{D}_4\right)\left(\mathcal{D}_3 + i\mathcal{D}_4\right)$$
$$+(B+2F) - B\sigma_3$$

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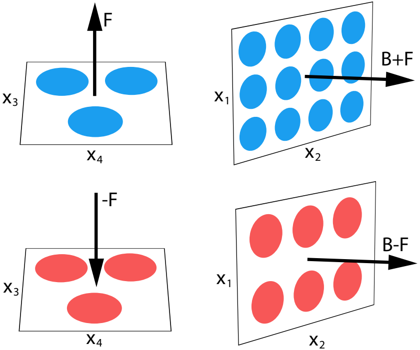
$+B - (B-2F)\sigma_3$

# Zero modes

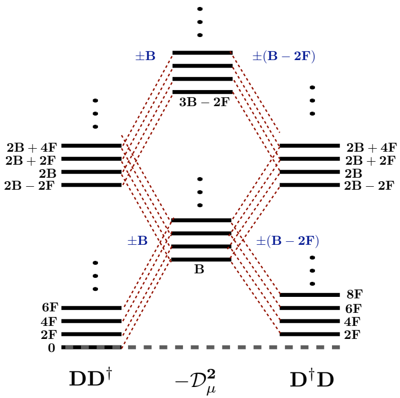
$$\tau = -1\,,\ \chi = -1\,,\ spin\uparrow,\ n_- = \frac{(B+F)}{2\pi}\,\frac{F}{2\pi}$$

$$\tau = +1\,,\ \chi = +1\,,\ spin\uparrow,\ n_+ = \frac{(B-F)}{2\pi}\,\frac{F}{2\pi}$$

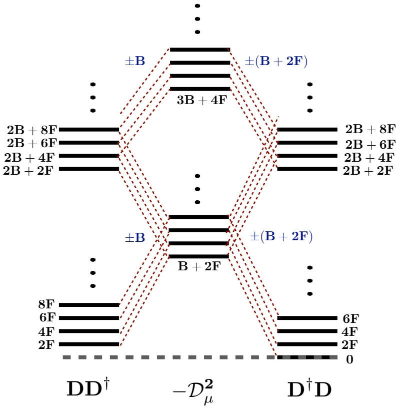
$$n_+ + n_- = B\frac{F}{2\pi^2} \quad , \quad n_+ - n_- = -\frac{F^2}{2\pi^2}$$



# Spectra




$\tau_3 = +1$




$\tau_3 = -1$

# Zero modes

$$B < F$$

$$\begin{array}{lrcl} \chi = +1 : & n_+ & = & 0 \\ \chi = -1 : & n_- & = & \begin{cases} \frac{(B+F)}{2\pi}\,\frac{F}{2\pi} \quad , \quad (\tau_3 = -1 \quad , \quad spin \uparrow) \\ \frac{(-B+F)}{2\pi}\,\frac{F}{2\pi} \quad , \quad (\tau_3 = +1 \quad , \quad spin \downarrow) \end{cases} \end{array}$$

$$n_+ + n_- = n_- = \frac{F^2}{2\pi^2} \quad , \quad n_+ - n_- = -n_- = -\frac{F^2}{2\pi^2}$$

$$n_\uparrow - n_\downarrow = \frac{BF}{2\pi^2}$$

# Quantization on $\mathbb{T}^4$

- nontrivial holonomy $\rightarrow$ topology
- twisted boundary conditions $\rightarrow$ fractional Pontryagin index (' t Hooft '81)
- zero twist $\rightarrow \exists$ quasi-abelian, covariantly constant, SD solutions (van Baal '96)

$$A_\mu(x_\nu + L_\nu) = \Omega_\nu^{-1}(x)(A_\mu(x_\nu) - i\partial_\mu)\Omega_\nu(x)$$

quantized flux for constant field strength (à la Aharanov-Bohm)

(van Baal '96, Al-Hashimi and Wiese '09)

$$\begin{aligned}
(B-F)\,L^2 &= 2\pi(N-M) \quad , \quad \text{for } \tau_3 = +1 \\
(B+F)\,L^2 &= 2\pi(N+M) \quad , \quad \text{for } \tau_3 = -1 \\
F\,L^2 &= 2\pi M \quad , \quad \text{for } \tau_3 = +1 \\
F\,L^2 &= 2\pi M \quad , \quad \text{for } \tau_3 = -1
\end{aligned}$$

# Quantization on $\mathbb{T}^4$

M: Instanton flux N:Magnetic flux

Zero modes:

$$B > F$$

index: $N_+ - N_- = -2M^2 = -\frac{F^2\,L^4}{2\pi^2}$

total number: $N_+ + N_- = 2MN = \frac{B\,F\,L^4}{2\pi^2}$

$$B < F$$

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# Dipole moments

$$\sigma_i^M = \frac{1}{2}\epsilon_{ijk}\langle\bar{\psi}\Sigma_{jk}\psi\rangle \quad , \quad \sigma_i^E = \langle\bar{\psi}\Sigma_{i4}\psi\rangle$$

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$$\sigma_3^M = \frac{1}{2}\langle\bar{\psi}\Sigma_{12}\psi\rangle \quad , \quad \sigma_3^E = \langle\bar{\psi}\Sigma_{34}\psi\rangle$$

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$$\Sigma_{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad , \quad \Sigma_{34} = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

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- $\sigma_3^M > \sigma_3^E$
- $\langle\bar{\psi}\Sigma_{34}\psi\,\bar{\psi}\Sigma_{34}\psi\rangle \approx \left(\frac{F}{2\pi^2 m^2 L^4}\right)\, B$

# Sphaleron rate (basics)

$$\Gamma_{CS} = \frac{(\Delta Q_5)^2}{Vt} = \int d^4x \left\langle \frac{g^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a(x) \frac{g^2}{32\pi^2} F^a_{\alpha\beta}\tilde{F}^{\alpha\beta}_a(0) \right\rangle$$

Diffusion of topological charge: $\frac{dN_5}{dt} = -c\, N_5\, \frac{\Gamma_{CS}}{T^3}$

- CP odd effects in QCD (CME)
- Baryon number (B+L) violation in E.W.

Weak coupling: $\Gamma_{CS} = \kappa\, g^4 T \log(1/g)\,(g^2 T)^3$ (Bödeker '98)

Strong coupling: $\Gamma_{CS} = \frac{(g^2 N)^2}{256\pi^3} T^4$ (Son, Starinets '02)

# Sphaleron rate (holography)

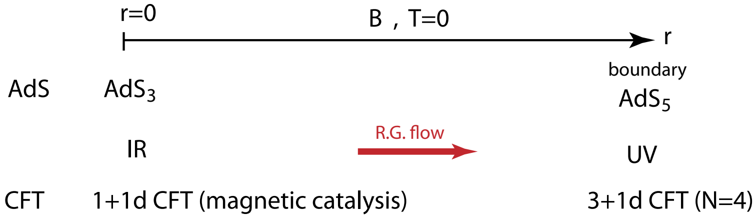
5-d Einstein-Maxwell-(Chern Simons):

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}(R+F^{MN}F_{MN}-\frac{12}{l^2})+\frac{1}{6\sqrt{3}\pi G_5}\int A\wedge F\wedge F$$
$$F = B dx_1 \wedge dx_2$$

$\exists$ solutions of the form: (D'Hoker, Kraus '08-'11)

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}(dx_1^2 + dx_2^2) + e^{2W(r)}dx_3^2$$

with: $U(r_h) = 0 \quad , \quad e^{2V(r)}, e^{2W(r)}|_{r\to\infty} \to r^2 \ (AdS_5)$,



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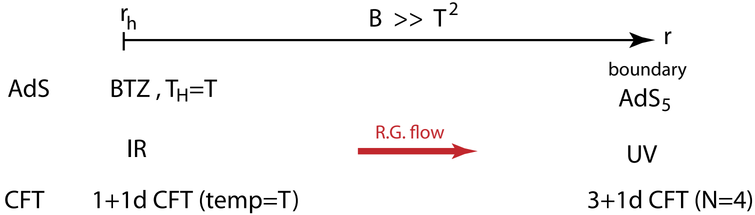

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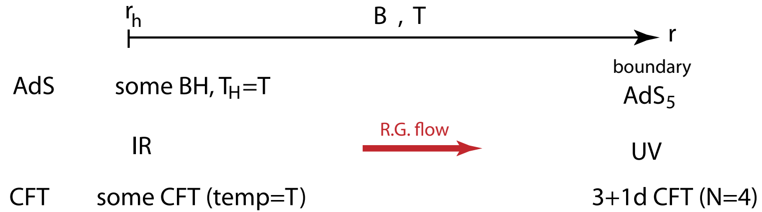

# Sphaleron rate (holography)

AdS/CFT: (axion) $\phi \Leftrightarrow \mathrm{tr}(F\tilde{F})$ (topological charge)

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FDT: $G^{sym}_{F\tilde{F},F\tilde{F}}(\omega,\vec{k}) = \coth(\frac{\omega}{2T})\,\mathcal{I}m G^R_{F\tilde{F},F\tilde{F}}(\omega,\vec{k})$

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$$\Rightarrow \quad \Gamma_{CS} = -\left(\frac{g^2}{8\pi^2}\right)^2 lim_{\omega\to 0}\frac{2T}{\omega}\mathcal{I}m\left[G^R_{F\tilde{F},F\tilde{F}}(\omega,\vec{k}=0)\right]$$

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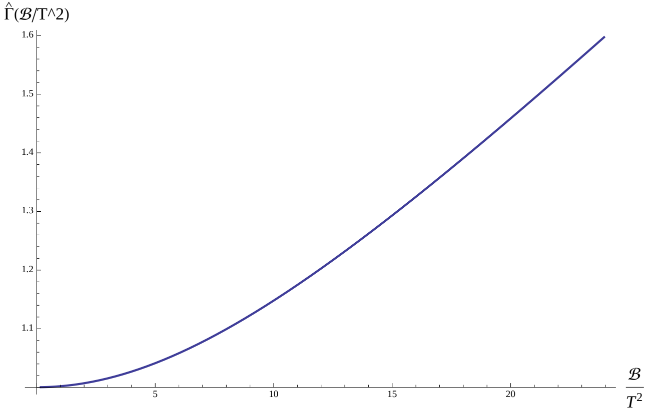


$$\Gamma_{CS} = \begin{cases} \frac{(g^2N)^2}{256\pi^3}\left(T^4 + \frac{1}{6\pi^4}B^2 + \mathcal{O}(\frac{B^4}{T^2})\right) & , \quad B << T^2 \\ \\ \frac{(g^2N)^2}{384\sqrt{3}\pi^5}\left(B\,T^2 + 15.9\,T^4 + \mathcal{O}(\frac{T^6}{B})\right) & , \quad B >> T^2 \end{cases}$$

# Sphaleron rate (holography)

$$B >> T$$

$$\Gamma_{CS} = \frac{(g^2 N)^2}{384\sqrt{3}\,\pi^4}\,\frac{B}{\pi}\;\times\;T^2$$

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- zero modes are magnetically confined in lowest Landau levels

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- zero modes are magnetically confined in lowest Landau levels
- strong interaction → back-reaction of B into the sphaleron
- spherical symmetry → axial symmetry

# An analogy

Electroweak sphaleron (Klinkhamer, Manton ,'89)

- Spherically symmetric for $SU(2)$ ($\Theta_W = 0$)
- Axially symmetric for $SU(2) \times U(1)$ ($\Theta_W \neq 0$)
- Has a magnetic dipole moment for $\Theta_W \neq 0$

# Conclusions & speculations

- Instanton + magnetic field has a rich structure
- Electric and magnetic dipole moments
- Zero modes play a crucial role
- $1^{\text{st}}$ order derivative expansion captures some lattice results
- Confinement ? (instantons with nonzero holonomy)
- At strong coupling:
  - Magnetic field always increases the sphaleron rate
  - Back-reaction of magnetic field into non-abelian sector
  - Strong magnetic field leads to dimensional reduction
- Weak coupling? (Diamagnetic response of zero modes?)