Large-angle Motion of a Simple Pendulum
(Sample writeup using Mathcad)

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Abstract

The large amplitude behavior of a simple bifilar pendulum, with amplitudes up to 45 degrees, is studied with the help of a computer-interfaced photogate. The zero-amplitude period is is deduced using interpolation and least-squares fitting methods. The results are compared with a complete theoretical treatment of a large amplitude pendulum. The experiment is found to be in poor agreement with theory, with systematic discrepancies of approximately 0.5% in the results for the small-amplitude period. These discrepancies greatly exceed the estimated uncertainty of 0.11%. A likely source of systematic error was found in the mechanical setup of the pendulum.

Theory

Using conservation of energy, and making substitutions as in the lab writeup, an expression for the large amplitude period of a simple pendulum can be obtained. We relate the period to that of a small-amplitude pendulum,

$\tau_0 := 2\cdot\pi\cdot\sqrt{\frac{l}{g}}$, where

$g \equiv 9.8039$ m/s^2, the acceleration of free fall, and

$l \equiv 0.547$ m, the length of the pendulum, as measured to the CM of the bob (see the analysis below for more discussion of this measurement).

In the above, the acceleration of gravity at Storrs was estimated using a standard forumula listed in the *Handbook of Geophysics and the Space Environment*, Air Force Geophysics Laboratory, 1985. Also, we have used MathCad's optimization feature to simplify this expression for more efficient use in later calcuations. The red asterisk indicates that a simplified result was successfully found. In this instance, double-clicking on the asterisk reveals that the optimization yielded a specific numerical result, which is subsequently used in place of the original functional form.

Defining $k(a) := \sin\left(a\cdot\frac{\pi}{360}\right)$, where *a* is the angular amplitude of oscillations measured in degrees,

the final expression for the calculated period is

$$\tau\_theory(a) := \tau_0\cdot\frac{2}{\pi}\cdot\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-(k(a)\cdot\sin(x))^2}}\,dx \quad .$$

Experimental procedure and results

The pendulum was released from heights calculated to yield angular amplitudes of 5,10, ..., 45 degrees. Its period was measured with a photogate, interfaced with an Apple IIe computer. Five independent launches for each amplitude were performed in order to estimate experimental uncertainties. The measurements yielded the following results, in seconds:

Trial

τ_experiment :=

	0	1	2	3	4
0	1.494	1.494	1.494	1.494	1.494
1	1.494	1.494	1.494	1.494	1.494
2	1.498	1.497	1.499	1.498	1.498
3	1.503	1.502	1.502	1.503	1.503
4	1.508	1.508	1.508	1.508	1.508
5	1.517	1.517	1.517	1.517	1.517
6	1.524	1.524	1.523	1.523	1.524
7	1.534	1.533	1.533	1.536	1.534
8	1.545	1.545	1.543	1.544	1.545

The row numbers index the amplitudes, whose values are specified here:

a_experiment :=

	0
0	5
1	10
2	15
3	20
4	25
5	30
6	35
7	40
8	45

degrees of arc

To analyze the data, start by defining range variables i and j for the data arrays, to index the trial number and the amplitude,

$i := 0 .. 8 \qquad j := 0 .. 4$.

Now calculate average experimental periods τ_aver for each initial amplitude:

$$\tau\_aver_i := \frac{1}{5} \cdot \sum_{j=0}^{4} \tau\_experiment_{i,j}$$

$$\tau\_aver^T = (1.494 \quad 1.494 \quad 1.498 \quad 1.503 \quad 1.508 \quad 1.517 \quad 1.524 \quad 1.534 \quad 1.544)$$

Here the vector was transposed to take less space on the page. As it turns out, the results are so reproducible that timing measurement errors play a negligible role in the overall uncertainty budget, and we will neglect them in the remainder of the analysis.

Before making a plot, fit the data to a second order polynomial, since the second-order term is the leading term in the expansion of the integral:

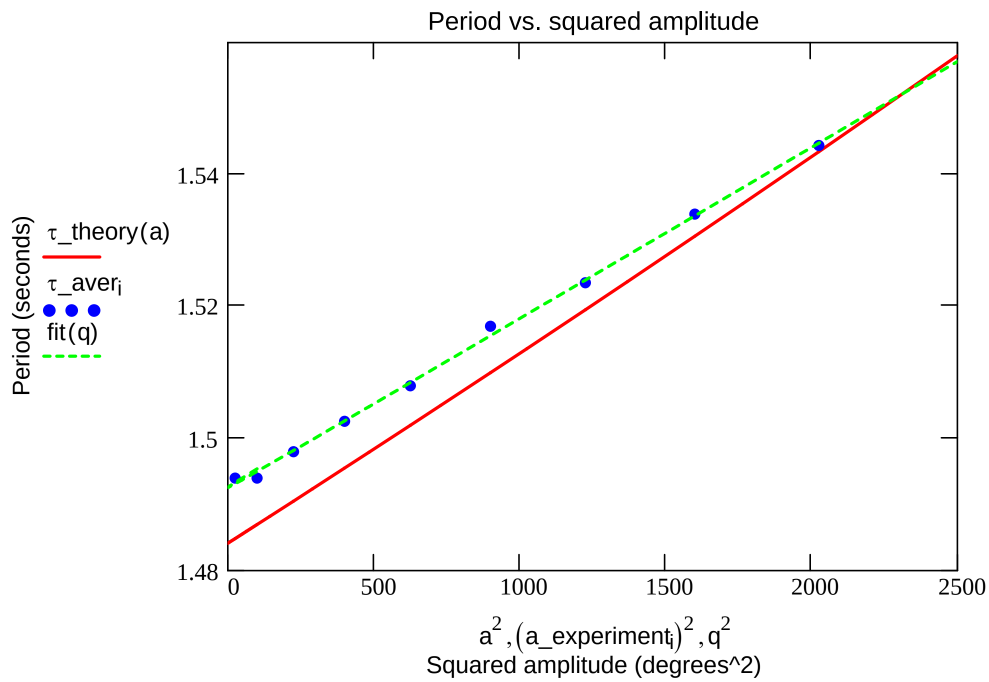
$z := \text{regress}(\text{a\_experiment}, \tau\text{\_aver}, 2) \quad t(x) := \text{interp}(z, \text{a\_experiment}, \tau\text{\_aver}, x)$

Now compare numerical values of the "zero-amplitude" periods we obtain from the theory and as extrapolated from our data.

$\text{fit}(0) = 1.493 \qquad \tau\text{\_theory}(0) = 1.484 \qquad \frac{\text{fit}(0) - \tau\text{\_theory}(0)}{\tau\text{\_theory}(0)} \cdot 100 = 0.57$ %.

They differ by ~0.6%.

The error seems to be systematic, which is apparent from a plot:



We can also calculate the "zero-amplitude" period by using the theoretical expression to find a result from each set of measurements, and then averaging together these nine results:

$$\tau_{avg} := \frac{1}{9} \cdot \left[\sum_{i=0}^{8} \left(\tau\text{\_aver}_i \cdot \frac{\tau_0}{\tau\text{\_theory}(\text{a\_experiment}_i)} \right) \right]$$

$\tau_{avg} = 1.48987$ seconds.

In principle we ought to have weighted this average to reflect the fact that the original data points had equal uncertainties, but they were then multiplied by the calculated ratio of τ_0 to the finite-amplitude period, making the uncertainties unequal. However, this makes almost no difference, since the zero-amplitude periods differ by only a few percent from the finite-amplitude periods. For future reference, the correct approach is to weight the data as indicated in the books by Taylor and Bevington,

$$\tau\_min\_var := \frac{\sum_{i=0}^{8}\left(\tau\_aver_i \cdot \frac{\tau\_theory(a\_experiment_i)}{\tau_0}\right)}{\sum_{i=0}^{8}\left[\left(\frac{\tau\_theory(a\_experiment_i)}{\tau_0}\right)^2\right]}$$

$\tau\_min\_var = 1.48981$ seconds, almost identical to the unweighted average.

Note that we had to change MathCad's default format for the results to see more decimal places than usual, to reveal the small difference between the two averages.

The corresponding mean-squared deviation for experimental measurements of τ is, ignoring weighting,

$$\sigma := \sqrt{\frac{1}{9}\cdot\sum_{i=0}^{8}\left(\tau\_aver_i - \tau\_min\_var\cdot\frac{\tau\_theory(a\_experiment_i)}{\tau_0}\right)^2}$$

which evaluates to $\sigma = 2.46 \times 10^{-3}$ seconds, the uncertainty for a single measurement.

The statistical uncertainty in the small-amplitude period is smaller by a factor of $N^{1/2}$, or 3 (taking into account that the sum in the denominator of Eq. (12) in the writeup is close to 1, and can be ignored for purposes of estimating the uncertainty.) The calculated uncertainty is thus, in percentage form,

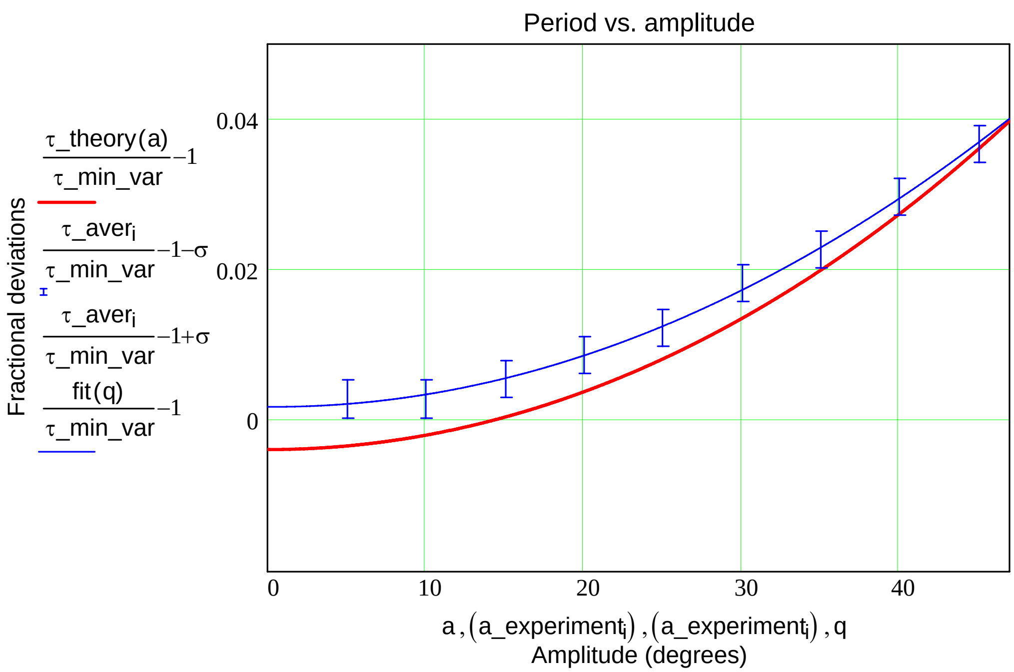
$$\frac{\sigma}{\tau\_min\_var}\cdot\frac{100}{3} = 0.06 \ \%.$$

Another source of uncertainty in τ_min_var is the uncertainty of the measured string length, a systemati error source. If we assume an uncertainty of 1 mm, the corresponding fractional change in τ_min_var is 0.09%. Thus the total estimated uncertainty is the quadrature sum of the two uncertainties, giving an overall fractional uncertainty of 0.11%.

The least-squares result for the small-amplitude period is somewhat larger than the regression result, by about two standard deviations. Again we can compare with the theoretical value:

$$\frac{\tau\_min\_var - \tau\_theory(0)}{\tau\_theory(0)} \cdot 100 = 0.38 \qquad \%.$$

The plot below shows a comparison of the measured (blue) and calculated (red) periods with the calculated zero-amplitude period, t_min_var. The solid blue curve is the linear regression fit to the data. The error bars for the experimental points are based on σ as calculated above :



Conclusions

Comparing the theoretical and experimental results in the plot above, we conclude that the theoretical value for the small-amplitude period disagrees systematically with the experimental result, especially at small amplitudes. The experimental value for τ_min_var differs from theory by 3-5 standard deviations, depending on the method of analysis used. Because the calculation of σ depends on the implicit assumption that there are no systematic deviations, the estimated uncertainties are not even particularly meaningful.

A likely explanation is that systematic deviations of the period resulted from the variable length of the pendulum at different amplitudes. The string of the pendulum is passing through a hole in the support bar that has diameter much larger than that of the string. As a result, at small oscillation amplitudes the string does not touch the lower end of the hole. In this case length of the string is from the top of the support bar to the C.M. of the bob. In the opposite case of large amplitude, the string is touching the lower end of the hole for most of the time. At intermediate amplitudes the string contacts the lower and of the hole for some fraction of the period of oscillation. As seen from the plot and by direct calculation for the largest-amplitude data point,

$$\frac{\tau\_aver_8 - \tau\_theory\left(a\_experiment_8\right)}{\tau\_theory\left(a\_experiment_8\right)} \cdot 100 = 0.06 \qquad \%,$$

the discrepancy at large amplitudes, for which the measured value of the pendulum length is valid, is much smaller than at small amplitudes. A change in the effective length of the string by 0.5 cm, or 1%, would change the small-angle period by 0.5%, about the size of the observed discrepancies. This possible problem with the apparatus was identified only in the late stages of data analysis, and so cannot easily be corrected immediately, but it can easily be resolved before the apparatus is used for future measurements.

Note

Subsequent to analyzing the data observed by a pair of students to prepare this sample report, we received another report that showed much smaller discrepanices. Thus it is possible that additional systematic errors were present in the data analyzed here, and that not all of the problem arose from ambiguities in the effective string length. In this connection it is worth noting that if the pendulum length were changed to 0.552 m, theory and experiment would agree almost within the error bars, although a systematic discrepancy would still be obvious in the plots.