312. Homework 5

R. Côté — Due: Thursday, March, 2004

PROBLEM 1: Bessel functions. In Arfken and Weber, do problem 11.5.5.

PROBLEM 2: In scattering theory, elastic cross sections are given by the expression

$$\sigma_{\text{elas.}} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2[\delta_l(k)] \ ,$$

where $E = \hbar^2 k^2/2\mu$ is the relative energy of the colliding particles of reduced mass μ, and $\delta_l(k)$ is the phase shift corresponding to the l^{th} partial wave. In this problem, we will find an expression for calculating $\delta_l(k)$. We consider central potentials, i.e. the interaction potential energy between two colliding particles depends only on the relative distance r: $V(\vec{r}) = V(r)$.

(i) Consider the radial differential equations for the scattering by two reduced potentials $U(r)$ and $\overline{U}(r)$

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - U(r)\right] u_l(r) = 0 \ , \tag{1}$$

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \overline{U}(r)\right] \overline{u}_l(r) = 0 \ . \tag{2}$$

Assuming that both $U(r)$ and $\overline{U}(r)$ tend to zero faster $1/r$ at large distances, write down the asymptotic functions $u_l(r)$ and $\overline{u}_l(r)$. Consider them energy normalized, i.e. that both functions possess a multiplicative factor $1/k$.

(ii) From the expression for the Wronskian

$$W[u_l(r), \overline{u}_l(r)] = u_l \overline{u}_l' - u_l' \overline{u}_l \ ,$$

where the prime denotes a derivative with respect to r, show that

$$\frac{d}{dr} W[u_l(r), \overline{u}_l(r)] = -(U - \overline{U}) u_l \overline{u}_l \ .$$

(Hint: use the above differential equations.)

(iii) Integrating the above expression, you will get

$$W[u_l(r), \overline{u}_l(r)]\big|_a^b = -\int_a^b dr\ \overline{u}_l(r)[U(r) - \overline{U}(r)] u_l(r) \ .$$

Choosing $a = 0$ and $b = \infty$, and with the boundary conditions $u_l(0) = \overline{u}_l(0) = 0$, show that

$$\tan\delta_l(k) - \tan\overline{\delta}_l(k) = -k\int_0^{\infty} dr\ \overline{u}_l(r)[U(r) - \overline{U}(r)] u_l(r) \ ,$$

provided that $U(r)$ and $\overline{U}(r)$ tend to zero faster than $1/r$ when $r \to \infty$, and are not more singular than $1/r^2$ at the origin.

(iv) Setting $\overline{U}(r) = 0$, show that

$$\tan\delta_l(k) = -k \int_0^\infty dr\; r\; j_l(kr)\; U(r) u_l(k,r)\;,$$

where $u_l(k,r) \to r[j_l(kr) - \tan\delta_l\; n_l(kr)]$ at large distances. This is the integral expression for the phase shift δ_l.

PROBLEM 3: Bessel functions. In Arfken and Weber, do problem 11.7.21.

PROBLEM 4: Legendre functions. In Arfken and Weber, do problems 12.1.3, 12.1.9, and 12.2.7.

PROBLEM 5: More Legendre functions. In Arfken and Weber, do problems 12.3.3, 12.3.11, and 12.3.20.