Preliminary Exam: Statistical Mechanics
Tuesday August 25, 2015, 9:00-12:00

Answer a total of any **THREE** out of the four questions.

For your answers you can use either the blue books or individual sheets of paper.

If you use the blue books, put the solution to each problem in a separate book.

If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set.

Be sure to put your name on each book and on each sheet of paper that you submit.

If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Possibly Useful Information

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \;\exp(-\alpha x^2 \;+\; \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx \; x \;\exp(-\alpha x^2) = \frac{1}{2\alpha}$$

I. Molecular oxygen is a paramagnetic gas. Each molecule has a spin angular momentum $\vec{S}$ with $|\vec{S}| = 1$. For a dilute gas (with N molecules in volume V) immersed in a magnetic field $\vec{B} = B\hat{z}$ along a z-axis, we can write the (magnetic) contribution to the Hamiltonian as follows:

$$H_{mag} = -\gamma \sum_{i=1}^{N} (S_{iz} B).$$

(Here γ is a constant.)

(a) If this (lattice) gas is at temperature T, calculate the probabilities for S_{iz} to have values +1, 0 and -1.

(b) Calculate (i) the magnetization $\vec{m} = \vec{M}/V$ (average magnetic moment per unit volume) and (ii) the entropy at temperature T and $B \neq 0$.

(c) For $B \neq 0$, find the limiting values of the entropy and magnetization as (i) $T \to 0$ and (ii) $T \to \infty$. Give a physical explanation for each of the limiting values you obtain.

II. Consider a classical ideal gas with fixed particle number N such that $pV = Nk_BT$ with a constant heat capacity with fixed volume of $C_V = Nk_B\eta$, where η can be considered to be constant (for the temperatures T we are interested in). Use Maxwell relations etc. to find the heat capacity at constant pressure, C_p. Then show that in that case the entropy can be written as

$$S = Nk_B \ln \frac{V}{N} + f(T) + \text{const.},$$

where $f(T)$ is a function of temperature and "const." does not depend on either temperature or volume. What is $f(T)$? Show that for an adiabatic process, both VT^η and pV^γ are constant. Here $\gamma = \frac{C_p}{C_V}$.

III. Consider an ideal gas of photons in 2-dimensions confined to an area A. The total energy is obtained by using the harmonic oscillator quantization (without the zero-point energy) as $\sum_{\vec{k}} n_{\vec{k}}\, \hbar\omega_{\vec{k}}$ where $n_{\vec{k}} = 0, 1, 2.....$ (being the number of photons); $\omega_{\vec{k}}$ is the angular frequency of an oscillator associated with the radiation mode (polarization index has been suppressed for clarity). The partition function Z is given by

$$\ln Z = -2 \sum_{\vec{k}} \ln\{1 - \exp(-\beta\hbar\omega_{\vec{k}})\}.$$

(a) Find the average occupation number $\left\langle n_{\vec{k}} \right\rangle$ and an integral expression in ω for the internal energy $U(T)$ at temperature T. Hence obtain the Planck's distribution (or radiation law) of energy density $\rho(\omega, T)$ in 2-dimensions. How does $\rho(\omega, T)$ behave as $\omega \to 0$ and $\omega \to \infty$ at fixed T?

(b) Evaluate the temperature dependence of the specific heat and find the pressure of this photon gas in terms of U. (You may express your answers in terms of the integrals $C_n = \int_0^\infty dx \, \frac{x^n}{e^x - 1}$.)

IV. A ferromagnetic XY model consists of unit classical spins, $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z) = (\cos\phi_i, \sin\phi_i, 0)$ such that $|\mathbf{S}_i| = 1$ on a three-dimensional cubic lattice with i labeling the site. The spins can point in any direction in the xy-plane. The Hamiltonian with nearest neighbor interactions is given by

$$H = -\frac{1}{2} J \sum_{i,\ell} \mathbf{S}_i \cdot \mathbf{S}_{i+\ell} - \mathbf{h} \cdot \sum_i \mathbf{S}_i,$$

where $\mathbf{h}$ is a field pointing in the x-direction (i.e., $\mathbf{h} = (h, 0, 0)$) and with i running over all the sites and ℓ running over the six nearest neighbors.

(a) For the noninteracting case $J = 0$, calculate the susceptibility $\chi = \left(\frac{\partial m_{ix}}{\partial h}\right)_{h=0}$ per spin showing that the average $\mathbf{m}_i = \langle \mathbf{S}_i \rangle$ reduces to a component m_{ix} along the x-axis. *Hint: While it is hard to evaluate the integral expression for* $\mathbf{m}(\mathbf{h})$, *it should be easy to evaluate its derivative at* $h = 0$.

(In order not to get confused with the nomenclature, rename $\mathbf{m}$ for this $J = 0$ case as $\mathbf{m_0}$, and χ as χ_0 before part (b).)

(b) Use your result in (a) to calculate the (critical) transition temperature in the ferromagnetic case using the standard "mean field theory" for $h = 0$ and $J \neq 0$. In this regard, find H_i such that (in mean field theory)

$$H = \sum_i H_i = -\mathbf{h}'(\mathbf{m_0}) \cdot \sum_i \mathbf{S}_i + \text{const.}$$

What is $\mathbf{h}'(\mathbf{m})$?

Hint: In order to find the critical temperature, repeat the procedure from part (a) to find χ, but self-consistently replace $\mathbf{h}'(\mathbf{m_0})$ by $\mathbf{h}'(\mathbf{m})$. Then find $\chi = \left(\frac{\partial m_{ix}}{\partial h}\right)_{h=0}$ and calculate the critical temperature T_C where $\chi \to \infty$.