Preliminary Exam: Statistical Mechanics, Tuesday January 9, 2018. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a separate blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Some possibly useful information:

$$\text{Sterling's asymptotic series}: \quad \ln N! \approx N \ln N - N + \frac{1}{2} \ln[2\pi N] \quad \text{as} \quad N \to \infty,$$

$$\int_0^\infty dx\ x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}, \qquad \int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2\ +\ \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0.$$

1. Consider a classical relativistic ideal gas with a fixed number $N \gg 1$ of free, massless, identical particles occupying a volume V. For a massless particle of momentum $\mathbf{p}$ the energy is $E_p = c|\mathbf{p}|$.

   (a) Derive expressions for the partition function $Z(T, V, N)$, the thermal energy U, and the enthalpy $H = U + PV$.

   (b) Determine the constant pressure and constant volume heat capacities c_P and c_V.

   (c) Derive expressions for the entropy S and the Helmholtz free energy $F = U - TS$, and explicitly verify the relation

   $$S(T, V, N) = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

   where V and N are held fixed.

2. Consider a classical ideal nonrelativistic gas of identical atoms in thermal equilibrium at a temperature T, with the gas being confined by a neutral atom trap with potential $V(r) = \gamma r^2$. The number of atoms, N, is fixed and large.

   (a) Determine the partition function $Z(N, T)$ and the entropy $S(N, T)$ of the system.

   (b) With the system being kept insulated so that no heat is being exchanged with the surroundings, the potential is lowered in a reversible way by decreasing the constant γ from an initial value γ_0 to a final value γ that is less than γ_0. If the temperature was initially T_0, what is the final temperature T?

3. Atomic particles of a non-thermal classical gas collide with electrons. In every collision an atom can lose or gain a small amount of energy ϵ (ϵ is a positive constant) with probabilities p or $q = 1 - p$ respectively. The time interval between two consecutive collisions is τ. An ensemble of N mono-energetic atomic particles with initial total energy NE_0 begins collisions with electrons at time $t = 0$ with collision energy change $\epsilon \ll |E_0|$.

   (a) Find the discrete energy distribution function $\rho(E, t)$ of atomic particles as a function of the atomic particle total energy E ($-\infty < E < +\infty$) at time $t = n\tau$. Show that the energy distribution $\rho(E, t)$ is described by a Gaussian function when the number of collisions is large, i.e. $n = t/\tau \gg 1$.

   Hint: Consider the atomic particle energy evolution as a random walk in the energy space E and derive the discrete binomial distribution for the atomic particle energy E as a function of the number of collisions $n = t/\tau$. In the $n \gg 1$ limit one can use a continuum approximation and apply Sterling's asymptotic formula.

   (b) Write down the differential equation that describes the time evolution of the energy distribution $\rho(E, t)$ at large time $t \gg \tau$. Show that this equation is the diffusion equation in the energy space $\{E\}$, and show that the Gaussian distribution derived in **(a)** satisfies this equation.

4. The equation of state of a non-ideal gas with N atoms is given by the expression:

$$P = \frac{Nk_BT}{V} + \frac{Ae^{-T/T_0}}{V^2},$$

   where T is the gas temperature, A and T_0 are positive constants, and k_B is the Boltzmann constant.

   (a) Find the work done in an isothermal, reversible gas expansion from an initial volume V_0 to a final volume $V_f = nV_0$.

   (b) Calculate the entropy change $\Delta S = S_f - S_0$ of the gas and the change in the internal energy $\Delta U = U_f - U_0$ in this expansion.