Preliminary Exam: Statistical Mechanics, Tuesday January 10, 2017. 9:00-12:00

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book and put the number of the problem on the front of the book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded. **Some possibly useful information**:

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty, \qquad \int_0^\infty dx\; x\; \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$\int_{-\infty}^{+\infty} dx\; \exp(-\alpha x^2 \;+\; \beta x) = \sqrt{\frac{\pi}{\alpha}}\exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0$$

1. A classical ideal gas of N non-interacting atoms at temperature T occupies a cubic volume $V = L^3$. At the time $t = 0$ a small hole is opened in the middle of one of the walls of the volume. The area of the hole is $A_{\mathrm{hole}} \ll L^2$. Outside the volume is unconfined empty space.

   (**a**) How many atoms leave the volume per unit time?

   (**b**) After what time has the pressure inside the volume decreased by $1/e$?

   (**c**) Calculate the average kinetic energies of the atoms inside and outside the volume, and determine the ratio $\langle E_{\mathrm{kin}}\rangle_{\mathrm{inside}}/\langle E_{\mathrm{kin}}\rangle_{\mathrm{outside}}$.

2. Consider the canonical ensemble of N non-interacting, spatially fixed (i.e. distinguishable) spins $S = \frac{1}{2}$ in a homogeneous magnetic field $\vec{B} = B\,\vec{e}_z$ which are described by the Hamiltonian operator

$$H = -\sum_{i=1}^{N} \vec{\mu}^{(i)} \cdot \vec{B}\,, \quad \mu_z^{(i)} = 2\,\mu_B\, S_z^{(i)}$$

   The states of the system $|m_s^{(1)}, m_s^{(2)}, \ldots, m_s^{(N)}\rangle$ satisfy $S_z^{(i)}| \ldots, m_s^{(i)}, \ldots\rangle = m_s^{(i)}| \ldots, m_s^{(i)}, \ldots\rangle$. Here $m_s^{(i)} = \pm\frac{1}{2}$, μ_B denotes the Bohr magneton, and $\vec{e}_z$ is a unit vector pointing in the z-direction..

   a. Determine the eigenenergies E_n of the Hamiltonian operator H, and their degeneracy d_n.

   b. Compute the partition function $Z(T, B) = \sum_{n=0}^{N} d_n \exp(-\beta E_n)$ with $\beta = 1/k_B T$.

   c. Compute the mean magnetization $\langle M\rangle = \langle 2\mu_B \sum_{i=0}^{N} S_z^{(i)}\rangle$, and
   (**i**) determine its value in the limit $T \to 0$, and
   (**ii**) derive its behavior at large temperatures $k_B T \gg \mu_B B$ (Curie's law).

3. A ideal gas of N non-interacting diatomic molecules in a one-dimensional space can be considered to be a system of N distinguishable simple harmonic oscillators each of frequency ω. The molecular mass of each molecule is m, the gas temperature is T and the gas occupies a fixed volume V.

(**a**) Calculate the partition function $Z_{\text{Q}}(T) = Z_{\text{Quantum}}(T)$ of the system if the vibrational motion of the molecules is quantum mechanical and the translation motion is classical.

(**b**) Derive a fully classical expression $Z_{\text{C}}(T) = Z_{\text{Classical}}(T)$ for the partition function of a gas of harmonic oscillators. Formulate conditions that are required for validity of this fully classical expression.

(**c**) Calculate the heat capacity of the harmonic oscillator gas using formulas derived in each of (**a**) and (**b**).

Hint: The heat capacity C is given by the formula $C = (\partial U/\partial T)_N$, where U is the internal energy of the system.

4. An ideal Bose gas of N spinless atoms occupies a volume V. The chemical potential μ (with $\mu \leq 0$) of the Bose gas can be found from the following expression for N:

$$N = \frac{1}{e^{-\mu/T} - 1} + V \frac{m^{3/2}}{2^{1/2}\pi^2\hbar^3} \int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{e^{(\epsilon-\mu)/T} - 1}$$

where ϵ and m are respectively the energy and mass of each of the atomic particles. T is the temperature of the Bose gas, and Boltzmann's constant has been set equal to one. The second term in the expression for N gives the number of atoms with energies $\epsilon > 0$.

(**a**) Derive an analytic expression for the critical temperature T_0 at which the chemical potential takes the value $\mu(T_0) = 0$ in the thermodynamic limit in which $V \to \infty$, $N \to \infty$ and $N/V \to$ constant.

Hint: The integral required for the derivation is given by

$$\int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{e^\epsilon - 1} = \frac{\pi^{1/2}}{2}\zeta(3/2)$$

where $\zeta(3/2) = 2.612$ is a Riemann zeta function.

(**b**) Calculate the number of atoms in translationally excited ($\epsilon > 0$) and ground ($\epsilon = 0$) states as functions of T/T_0, if the gas temperature T is below the critical value T_0.

(**c**) Rewrite the expression given above for N in the case of a two-dimensional Bose gas. In this rewritten expression do we need the first term in the limit $\mu \to 0$? Explain, why in the thermodynamic limit $V \to \infty$ and $\mu \to 0$ that Bose condensation is not possible for a two-dimensional Bose gas.