STATISTICAL MECHANICS

Preliminary Examination

Tuesday 01/14/2014

09:00 - 12:00 in P-121

Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

You are allowed to use a result stated in one part of a problem in the subsequent parts even if you cannot derive it. On the last page you will find some potentially useful formulas.

Problem 1. Consider a classical ideal gas of N particles in three dimensions confined to an external potential $V(r) = K(r/r_0)^\alpha$, where $K > 0$, $\alpha > 0$, and $r_0 > 0$ are constants.

(a) Show that the heat capacity is $C = \left(\frac{3}{2} + \frac{3}{\alpha}\right) Nk$.

(b) Suppose the shape parameter of the potential α may be varied. Why is it that the standard heat capacity of the free ideal gas $C_V = \frac{3}{2}Nk$ occurs in the limit $\alpha \to \infty$?

Problem 2. (a) For the two dimensional non-relativistic Bose gas with zero spin, calculate the chemical potential as a function of temperature and (area) density.

(b) Do we have critical density and do we need to add Bose-Einstein condensate at low temperatures, as in the case of three dimensional Bose gas?

Problem 3. For a non-relativistic ideal degenerate ($T = 0$) Fermi gas, find the average relative velocity $|\mathbf{v}_1 - \mathbf{v}_2|$ of two particles.

Problem 4. The dynamics of the vibrational normal modes of a solid made of N atoms can be approximated in terms of uncoupled harmonic quantum oscillators by the Hamiltonian $H = \sum_{j=1}^{3N} (p_j^2/(2m) + \frac{1}{2}\, m\, \omega_j^2 x_j^2)$.

(a) Calculate the canonical partition function $Z(T, N)$ of the system, determine its internal energy U, and show that it can be written as $U(T, N) = \int_0^\infty \frac{1}{2}\, \hbar\, \omega \coth(\frac{1}{2}\beta\, \hbar\, \omega)\sigma(\omega)\, d\omega$ with the normal-mode vibrational frequency distribution $\sigma(\omega) = \sum_{j=1}^{3N} \delta(\omega - \omega_j)$.

(b) Consider the Einstein model of a solid where $\sigma(\omega) = \sum_{j=1}^{3N} \delta(\omega - \omega_j)$ with $\omega_j = \omega_E\ \forall\, j$. Derive the expression for the heat capacity C. Show that C satisfies the Dulong-Petit law for $T \gg T_E$, and vanishes exponentially for $T \ll T_E$ where $T_E = \hbar\, \omega_E / k_B$.

(c) Consider the Debye model which assumes $\sigma(\omega) = 9N\omega^2/\omega_D^3$ if $\omega \le \omega_D$ and zero otherwise, where the value of ω_D is fixed by the normalization condition $\int_0^\infty \sigma(\omega)\, d\omega = 3N$. Derive the expression for the heat capacity C. Show that C satisfies the Dulong-Petit law for $T \gg T_D$, and vanishes like $C = \text{const} \times T^n$ for $T \ll T_D$ where $T_D = \hbar\, \omega_D / k_B$.

Remark: your calculation should determine the power n.

Hint: you may encounter an integral of the type $\int_0^\infty dx\ x^4\, e^x/(e^x - 1)^2 = 4\pi^4/15$.

(d) What does the third law of thermodynamics imply for the heat capacity? Do the models of Einstein and Debye discussed in part (b) and (c) satisfy the third law of thermodynamics?

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx \; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega}\right)$$