Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented *separately* in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

You are allowed to use a result stated in one part of a problem in the subsequent parts even if you cannot derive it. On the last page you will find some potentially useful formulas.
Problem 1. Let us define \( D = \frac{1}{2}(xp + px) \), where \( x \) is the position operator and \( p \) the momentum operator in one dimension.

(a) Calculate \([D, x^m]\) and \([D, p^n]\) where \( m \) and \( n \) are integers.

(b) Consider the Hamilton operator \( H = \frac{p^2}{2m} + V(x) \) with the potential \( V(x) = \alpha x^\beta \) where \( \alpha \) and \( \beta \) are real non-zero constants. Calculate \( U(\lambda)HU^\dagger(\lambda) \) with \( U(\lambda) = \exp(i\lambda D/\hbar) \).

(c) There is a value for \( \beta \) in the potential \( V(x) = \alpha x^\beta \) for which the Hamiltonian in part (b) transforms as \( U(\lambda)HU^\dagger(\lambda) = f(\lambda) H \). What is the function \( f(\lambda) \)?

Hints: Recall the identity for two non-commuting linear operators \( A \) and \( B \)

\[
\exp(\lambda A) B \exp(-\lambda A) = B + \frac{\lambda^1}{1!} [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \frac{\lambda^3}{3!} [A, [A, [A, B]]] + \ldots
\]

You may do the mathematics formally, ignoring issues such as the precise definitions and domains of various operators.

Problem 2. (a) Given the usual eigenstates \(|j, m\rangle\) of the angular momentum operators \( J^2 \) and \( J_z \), determine the expectation values \( \langle j, m|J_x|j, m\rangle \) and \( \langle j, m|J_y|j, m\rangle \).

(b) Find the standard deviation \( \Delta J_x = \sqrt{\langle j, m|J^2_x|j, m\rangle - \langle j, m|J_x|j, m\rangle^2} \), and \( \Delta J_y \) defined analogously.

(c) Determine the eigenvalues and construct the real eigenfunctions of the Hamiltonian involving orbital angular momentum of a single particle, \( H = a(L_x^2 + L_y^2) + bL_z^2 \), where \( a \neq b \) are real constants.

Possibly helpful identity: \( Y_{l,m}^*(\theta, \phi) = (-1)^m Y_{l,-m}^*(\theta, \phi) \).

Problem 3. A particle of mass \( m \) and electric charge \( q \) is constrained to move in a tightly confining ring of radius \( R \); call the remaining coordinate along the ring \( x \). The motion along \( x \) is free, i.e., there are no forces acting on the particle in the direction \( x \). Determine:

(a) Eigenvalues and eigenfunctions of energy.

(b) The maximum value of the electric current \( I \) in the first excited state.

Hint: The current density of a quantum particle is \( \mathbf{j} = \frac{i\hbar q}{2m} (\psi \nabla \psi^* - \psi^* \nabla \psi) \).
**Problem 4.** A particle with the energy $E$ and mass $m$ is scattered by the potential field

$$U(r) = U_0 (R/r) \exp(-r/R),$$

where $U_0$ and $R$ are positive constants. Calculate the scattering amplitude $f(E, \theta)$ ($\theta$ is the scattering angle) and the total scattering cross section $\sigma$ using the first Born approximation.

**Problem 5.** Consider the Hamiltonian

$$H = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| + V |2\rangle\langle 1| + V^*|1\rangle\langle 2|,$$

with $|V| \ll |E_2 - E_1|$.

(a) Find the eigenvalues of energy and the corresponding normalized eigenstates up to the lowest nontrivial order in the strength of the perturbation $V$. Denote these by $E_1', |1'\rangle$ and $E_2', |2'\rangle$, with $E_1' \to E_1$ as $V \to 0$ and so on.

(b) Suppose we are studying transitions from yet another state $|g\rangle$ to the states $|1\rangle$ and $|2\rangle$ governed by the operator $D$, and have the known transition matrix elements $\langle 1|D|g\rangle = d$, $\langle 2|D|g\rangle = 0$. At this level the transition $g \to 2$ is evidently forbidden. However, the perturbation $V$ leads to a small admixture of the original state $|1\rangle$ in the state $|2'\rangle$. Thus, a transition that to an observer unaware of the existence of perturbation $V$ might seem to be $g \to 2$ is possible after all. Find the corresponding matrix element $\langle 2'|D|g\rangle$. 

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\[ \ln N! \approx N \ln N - N \text{ as } N \to \infty \]

\[ \int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \text{ with } \Re(\alpha) > 0 \]

\[ \int_0^\infty dx \, x \exp(-\alpha x^2) = \frac{1}{2\alpha} \]

\[ a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + i \frac{p}{m\omega} \right) \]