QUANTUM MECHANICS

Preliminary Examination

Friday 08/23/2013

9:00am - 1:00pm in P-121

Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

On the last page you will find some potentially useful formulas.
Problem 1. A quantum particle is confined to the interval $[-a,a]$. It is described by the time-dependent wave function

$$
\psi(x,t) = \frac{1}{\sqrt{2a}} \{ \cos(\pi x/a) \exp[-i(\hbar \pi^2/8ma^2)t] - \sin(\pi x/a) \exp[-i(\hbar \pi^2/2ma^2)t] \}.
$$

(a) Show that $\psi(x,t)$ is properly normalized and find the associated probability current $J(x,t)$.

(b) Compute the probability $P_{\text{left}}(t) = \int_{-a}^{0} |\psi(x,t)|^2 dx$ for finding the particle in the left half of the interval.

(c) Compare the probabilities $P_{\text{left}}(0)$ and $P_{\text{left}}(\tau)$ where $\tau = \frac{4ma^2}{\hbar \pi}$ and show that $P_{\text{left}}(0) > P_{\text{left}}(\tau)$.

(d) Show that $P_{\text{left}}(0) - P_{\text{left}}(\tau) = \int_0^\tau J(0,t) dt$ and interpret this result.

Problem 2. The angular momentum operators for states with a certain $l$ value can be expressed in a matrix representation as

$$
L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
$$

The above matrix elements are defined with respect to an orthonormal basis set $\mathcal{B} = \{ |\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle \}$. For example, the $(i,j)^{th}$ element of $L_x$ is $\langle \phi_i | L_x | \phi_j \rangle$, with $\langle \phi_i | \phi_j \rangle = \delta_{ij}$ ($i, j = 1, 2, 3$).

(a) Identify the $l$ value and a basis $\mathcal{B}$ appropriate to this matrix representation.

(b) Compute the matrices $L_+, L_-$ and $L^2$ in your basis.

(c) For the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)$, compute the expectation values $\langle \psi | L_x | \psi \rangle$, $\langle \psi | L_y | \psi \rangle$ and $\langle \psi | L_z | \psi \rangle$. 

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**Problem 3.** For the oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$$

let us define the annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega \hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega \hat{x} - i\hat{p}).$$

(a) Prove that the commutation relation for these operators is

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

(b) Find the expression of the Hamiltonian operator $\hat{H}$ in terms of these operators.

(c) Prove that the eigenstate of the operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

has the form

$$\langle x|\alpha\rangle = C \exp\left(-\frac{1}{2}a^2 - \frac{1}{2}m\omega \frac{x^2}{\hbar} + \sqrt{\frac{2m\omega}{\hbar}} a x\right).$$

Find the normalization constant $C$ so that the states are normalized as

$$\langle \alpha|\alpha\rangle = \exp \alpha \alpha^*$$

(this is a bit unusual normalization, but very convenient).

(d) The vacuum state of the system corresponds to the zero eigenstate of the annihilation operator $\hat{a}|0\rangle = 0$, while the excited states are obtained by the action of the creation operator $(\hat{a}^\dagger)^n|0\rangle$ (up to normalization). Using the results of the previous questions, obtain the coordinate representation of the wavefunction of the first excited state of the oscillator.

**Problem 4.** The unperturbed Hamiltonian of a two-state system is given by

$$H_0 = E_1^0|1\rangle\langle 1| + E_2^0|2\rangle\langle 2|$$

The system is subjected to a time-dependent perturbation,

$$V(t) = \lambda \cos \omega t|1\rangle\langle 2| + \lambda \cos \omega t|2\rangle\langle 1|$$

where $\lambda$ is real.
(a) At $t = 0$ the system is in the first eigenstate $|1\rangle$. Using time-dependent perturbation theory, and assuming $E_1^0 - E_2^0$ is not close to $\pm \hbar \omega$, find the probability that the system is in state $|2\rangle$ at time $t$.

(b) Why is this procedure not valid when $E_1^0 - E_2^0 \sim \pm \hbar \omega$?

**Problem 5.** Consider two particles, each with spin 1/2. Using an explicit matrix representation of the system in the uncoupled basis ($S_1$, $S_2$ diagonal), find the eigenstates and eigenvalues of total spin, $S = S_1 + S_2$. What are the Clebsch-Gordan coefficients relating the original basis to the one where $S^2$ is diagonal?
\[
\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty
\]

\[
\int_{-\infty}^{+\infty} \! dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp \left( \frac{\beta^2}{4\alpha} \right) \quad \text{with} \quad \Re(\alpha) > 0
\]

\[
\int_0^\infty \! dx \, x \exp(-\alpha x^2) = \frac{1}{2\alpha}
\]

\[
a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + i \frac{p}{m\omega} \right)
\]