QUANTUM MECHANICS/ STATISTICAL PHYSICS

Preliminary Examination

August 28, 2009

9:00 - 15:00 in P-121

Answer a total of **SIX** questions, choosing **at least TWO** from Section A, and **the rest** form Section B. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

For a positive integer n and real α,

$$\int_0^\infty dx\, x^n\, \exp(-x) = n!, \quad \int_{-\infty}^{+\infty} dx\, \exp(-\alpha x^2) = \sqrt{\frac{\pi}{\alpha}}, \quad \int_0^\infty dx\, x\, \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

First few spherical harmonics

$$Y_{00} = \left(\frac{1}{4\pi}\right)^{1/2}, \quad Y_{10} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta, \quad Y_{1\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, \exp(\pm i\phi)$$

$$Y_{20} = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1), \quad Y_{2\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta\, \exp(\pm i\phi),$$

$$Y_{2\pm2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta\, \exp(\pm 2i\phi)$$

SECTION A - STATISTICAL PHYSICS

A1. The energy distribution function $f(E)$, normalized to unity, of an isotropic, *ideal* gas of excited atoms (with mass M) may be investigated experimentally by detecting the spectra of atomic emission. The frequency ν of the photons emitted by the moving atoms is different from that of the resting atoms ν_0. The frequency shift $(\nu - \nu_0)$ is calculated from the Doppler formula

$$\nu - \nu_0 = (V_z/c)\nu_0,$$

where V_z is the projection of the velocity of the atom along the direction of observation (of emission) and c is the speed of light.

The probability density function $\rho(\nu - \nu_0)$ describes the probability of observing the value $(\nu - \nu_0)$ for the Doppler shift.

(a) Derive an analytical expression for $\rho(\nu - \nu_0)$ via the energy distribution function $f(E)$. The final formula should only include the given parameters, ν, ν_0, c, the mass of an atom M and the known function $f(E)$.

(b) Calculate the function $\rho(\nu - \nu_0)$ for a gas of Fermions at zero temperature $(T = 0)$ with an energy distribution function $f(E) = (3/2)E^{1/2}/E_f^{3/2}$ where E_f is the Fermi energy.

(c) Calculate $\rho(\nu - \nu_0)$ for a classical gas in thermal equilibrium at temperature T.

A2. Consider a gas of molecular oxygen, which is paramagnetic. Each diatomic molecule has a spin angular momentum $\mathbf{S}$ with $|\mathbf{S}| = 1$. The magnetic dipole moment per molecule is therefore $\boldsymbol{\mu} = \gamma\mathbf{S}$. For a dilute gas (with N molecules in volume V) immersed in a magnetic field $\mathbf{B}$ along a certain direction z, we can write the magnetic induction term of the Hamiltonian as follows:

$$\mathcal{H}_{mag} = -\gamma \sum \mathbf{S}_i \cdot \mathbf{B} = -\gamma \sum_i BS_{iz}.$$

(a) If the gas is at temperature T, calculate the probabilities for S_{iz} to have the values +1, 0, and -1.

(b) Calculate the magnetization $\mathbf{M}$ = average magnetic dipole moment per unit volume.

(c) If the magnetic field is sufficiently weak, the magnetization is proportional to the field: i.e., $\mathbf{M} = \chi \mathbf{B}$. Calculate the susceptibility χ in this low field limit.

(d) Calculate the value of $\mathbf{M}$ in the limit that $B \to \infty$.

A3. A diatomic molecule may be described by three translational, two rotational, and one vibrational degrees of freedom. Suppose that a quantum mechanical solution yields a set of energy levels

$$E_{k,J,\nu} = E_k + BJ(J+1) + \hbar\omega(\nu + 1/2)$$

with $J = 0, 1, 2, ...$; $\nu = 0, 1, 2, ...$

(a) Identify the above terms and carefully explain *any underlying assumptions*. Write down an expression for the canonical partition function for an assembly of N such molecules contained in a volume V (at temperature T) using the above. Find the low temperature behavior of the heat capacity due to the rotational degrees of freedom.

(b) Obtain a closed expression for the vibrational heat capacity of such diatomic molecules. What are the high and low temperature asymptotic values of this heat capacity?

SECTION B - QUANTUM MECHANICS

B1. Consider a particle of mass m and charge q constrained to move on the curved surface of a nano-size cylinder with radius R and length L.

(a) Determine the integrals of motion, eigenfunctions, and eigenvalues which describe the quantum states of particle motion.

Using the lowest orders of perturbation theory, calculate:

(b) The ground state wavefunction and energy of the particle in a weak uniform electric field E applied along the direction (Z) of the axis of the cylinder.

(c) The shift of the ground state energy in a weak uniform magnetic field B applied along Z.

(The perturbative potential induced by a magnetic field may be written as

$$V = -(q/mc)\mathbf{A} \cdot \mathbf{p} + (q^2/2mc^2)A^2 = -\omega_c L_z + m\omega_c^2 \rho^2/2,$$

where L_z is the operator of projection of the angular momentum on the Z axis and $\omega_c = qB/mc$ is the cyclotron frequency of motion of the charged particle in a uniform magnetic field B.)

B2. The projection L_z of the angular momentum operator $\mathbf{L}$ is involved in the operator transformation

$$\exp(-i\pi L_x a)\ L_z\ \exp(i\pi L_x a)$$

where a is a positive constant such that $a \leq 1$.

(a) Demonstrate that the result of the above transformation is given by a linear function of the operators of angular momentum projection, L_z and L_y.

(b) Calculate the coefficients of this linear function and describe the geometrical and physical meaning of the above transformation.

B3. A planar rigid rotator having a moment of inertia I and an electric dipole moment **d** is placed in a homogeneous electric field **E**.

(a) By considering the electric field as a perturbation, determine the first non-vanishing correction to the energy levels of the rotator. Here the rotator is restricted to move in a plane.

(b) If the rigid rotator is not restricted to move in a plane, what are the eigenfunctions and eigenvalues of the unperturbed problem. In this case, calculate the first order correction to the energy levels under the above perturbation (due to the electric field).

B4. Consider the one-dimensional quantum scattering of an electron of energy E from a square barrier of height V_0 and width $2a$. Take the incident electron to be described by a plane wave.

(a) Calculate the transmission coefficient T through the barrier for $E > V_0$.

(b) Under what conditions is there 100 % transmission? Give a physical explanation.

(c) What is the limiting value of T as $E \to \infty$? Explain.

B5. This problem deals with photoionization of the hydrogen atom. Starting at $t = 0$, the atom in the ground state is acted on by a uniform oscillating electric field polarized along z: $\mathbf{E} = \hat{\mathbf{z}}\ E_0 \sin(\omega t)$. The frequency ω is just sufficient to ionize the atom.

(a) Using perturbation theory (i.e., the Golden Rule), calculate the ionization rate. Make the assumption that the electron in its final state is a free particle and can be described by a plane wave of wave vector $\mathbf{k}$, with $ka_0 \ll 1$. Here a_0 is the characteristic size of the ground state (the Bohr radius).

(b) How does the ionization rate vary with the angle θ between the E-field polarization direction ($\hat{\mathbf{z}}$) and the direction of the momentum of the free electron?

B6. (a) Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be spin operators of two identical spin-half particles. Write down the simultaneous eigenfunctions and eigenvalues of the operators $\mathbf{S}^2$ and S_z, where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and $S_z = S_{1z} + S_{2z}$. Note that these are the so-called singlet and triplet states. Prove that these are eigenfunctions of $\mathbf{S}_1 \cdot \mathbf{S}_2$ as well. What are the corresponding eigenvalues?

(b) For a system of two identical particles as in (a), interacting via a *time-independent* (spin) Hamiltonian that commutes with $\mathbf{S}^2$ and S_z, suppose the triplet and singlet energy levels are given by E_1 and E_2 with $E_1 > E_2$. If such a system is initially prepared to be in the state $|\uparrow_1, \downarrow_2\rangle$, how long will it take to be in the state $|\downarrow_1, \uparrow_2\rangle$, if at all? Explain.