QUANTUM MECHANICS/ STATISTICAL PHYSICS

Preliminary Examination

August 22, 2008

9:00 - 15:00 in P-121

Answer a total of **SIX** questions, choosing **at least TWO** from Section A, and **the rest** form Section B. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else.

SECTION A - STATISTICAL PHYSICS

A1. Consider a gas of free, nonrelativistic electrons, with uniform density ρ.

(a) Show that the Fermi energy is given by

$$k_F = \left(3\pi^2 \rho\right)^{1/3}.$$

Hence show that the total quantum mechanical energy of the electron gas at low temperature in a spherical volume of radius R is

$$E_{QM} = \frac{2\hbar^2}{15\pi m}\left(\frac{9}{4}\pi\, n\right)^{5/3}\frac{1}{R^2}$$

where m is the electron mass, and n is the total number of electrons.

(b) Consider a cold star (a white dwarf) with N nucleons, of mass M and of uniform density with radius R. Show that the gravitational energy is

$$E_{grav} = -\frac{3}{5}\frac{GN^2M^2}{R}$$

where G is Newton's constant.

(c) Use the results of (a) and (b) to show that for a white dwarf, there is a radius at which the repulsive quantum mechanical degeneracy pressure of the electrons balances the attractive gravitational pressure, and find an expression for this radius. Assume there are $q = 1/2$ electrons per nucleon.

A2. The Heisenberg model for ferromagnetism is defined by the Hamiltonian

$$\mathcal{H} = -J\sum_{\{i,j\}} \mathbf{S_i}\cdot\mathbf{S_j} - \mu\sum_i \mathbf{H}\cdot\mathbf{S_i}$$

where $\mathbf{S_i}$ is a three dimensional unit vector which lives on a square lattice in d-dimensions (the index i numbers the lattice sites). The interaction is only between the nearest neighbors on the lattice - the summation in the first term is over all pairs $\{i,j\}$ of the nearest neighbors. Consider $J > 0$ and let the external field $\mathbf{H}$ be uniform with $h = \mu|\mathbf{H}|$. Using the mean field approximation do the following:

(a) Find the critical temperature T_c (i.e., the temperature below which the system a has nonvanishing magnetization m even in a vanishing external field $h = 0$).

(b) Find critical exponent δ defined as $h \sim m^\delta$, at $T = T_c$.

(c) Find critical exponent β defined as $m \sim (T_c - T)^\beta$, at $h = 0$; $T < T_c$ (here T is close to T_c, so that the relevant expressions can be expanded in $(T_c - T)$).

Hint: In the mean field approximation, we approximate the interaction of each individual spin by an interaction with the "mean field"

$$\mathcal{H} \to -J \sum_i a\boldsymbol{\sigma_0} \cdot \mathbf{S_i} - \mu \sum_i \mathbf{H} \cdot \mathbf{S_i}.$$

Here a is the number of the nearest neighbors with which a given spin interacts, while σ_0 is the mean magnetization determined self consistently from requiring that

$$\boldsymbol{\sigma_0} = < \mathbf{S} > .$$

This mean field equation serves as the basis for the determination of the phase structure as well as of critical exponents.

A3. Consider a heteronuclear, diatomic molecule with moment of inertia I. (In this problem, consider only the rotational motion of the molecule.)

(a) Using classical statistical mechanics, calculate the specific heat of this system at temperature T.

(b) Using quantum statistical mechanics, find the expression of the partition function and the average energy of the system as a function of temperature.

(c) Derive an expression for the specific heat that is valid at very low temperature by simplifying your result in (b). What is the temperature range in which your expression is valid?

(d) Derive a high-temperature approximation to the specific heat from result in part (b). What is the range of validity of your approximation?

SECTION B - QUANTUM MECHANICS

B1. Consider a hydrogen atom placed in a uniform electric field of strength $\mathcal{E}$. Let the field point in the z direction, so that the perturbation energy is

$$H_I = e\,\mathcal{E}\,z.$$

(a) Show that the ground state energy is not affected by this perturbation, to first order in $\mathcal{E}$.

(b) Show that the perturbation does not mix states of different magnetic quantum number m_l.

(c) The first excited state of the hydrogen atom is 4-fold degenerate. In the usual notation,

$$\begin{aligned}
\psi_{200} &= \frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\left(1-\frac{r}{2a}\right)e^{-r/(2a)} \\
\psi_{211} &= -\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}\,r\,e^{-r/(2a)}\,\sin\theta\,e^{i\phi} \\
\psi_{210} &= \frac{1}{\sqrt{2\pi a}}\frac{1}{4a^2}\,r\,e^{-r/(2a)}\,\cos\theta \\
\psi_{21,-1} &= \frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}\,r\,e^{-r/(2a)}\,\sin\theta\,e^{-i\phi}.
\end{aligned}$$

Use degenerate perturbation theory to find the effect of the perturbation on the first excited state, to first order in $\mathcal{E}$. Sketch the resulting energy levels.

B2. Consider the Dirac equation

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-c\,\vec{\alpha}\cdot\vec{p} - \beta\,mc^2\right)\psi$$

where the Dirac matrices are expressed in terms of the 2×2 Pauli matrices $\vec{\sigma}$ and the 2×2 unit matrix $\mathbf{1}$ as

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

(a) Write down the charge and current densities and obtain the related conservation law that is satisfied.

(b) Using the Pauli matrix relations $\sigma^j \sigma^k = i\epsilon^{jkl}\sigma^k$, show that for any 3-vectors $\vec{D}$ and $\vec{G}$

$$(\vec{\alpha} \cdot \vec{D})(\vec{\alpha} \cdot \vec{G}) = \vec{D} \cdot \vec{G} + i\vec{\sigma}\,' \cdot \left(\vec{D} \times \vec{G}\right)$$

where

$$\vec{\sigma}\,' \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

(c) Include the interaction with a static magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Use the result of part (b) to show that the Dirac equation can now be written as

$$\left(E^2 - \left(c\vec{p} - e\vec{A}\right)^2 - m^2c^4 + e\hbar c\,\vec{\sigma}\,' \cdot \vec{B}\right)\psi = 0$$

where the time dependence of the solution is $\psi(x,t) = e^{iEt}\psi(x,0)$. Hence, in the nonrelativistic limit, show that one obtains the 2×2 Pauli Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}$$

and comment on the significance of the numerical coefficient in front of the last term.

B3. A particle of mass m is in the ground state of a one dimensional harmonic oscillator potential of frequency ω. At time $t = 0$, the frequency of the potential is instantaneously increased threefold to 3ω.

(a) What is the probability at time $t = 0$ to find the particle in the second excited state ($n = 2$) of the new potential?

(b) Prove that the probability to find the particle in a state with any odd n of the new potential is strictly zero.

(c) Starting at time zero, the particle moves in the new potential. Then at time $t = 4\pi/\omega$, the potential is switched back to the original one with frequency ω. What is the probability that the particle is in the ground state of the new potential at time $T = 5\pi/\omega$?

B4. In one dimensional quantum mechanics, consider the two operators

$$A = \frac{1}{2}(p^2 + x^2)$$

and

$$B = \frac{1}{2}(xp + px).$$

(a) Show that the operator A acts as the rotation generator in the phase space, namely that under its action, the vector (p, x) rotates like a vector on a plane.

(b) Show that the operator B generates a scaling transformation under which p and x are scaled by the inverse factors $p \longrightarrow \alpha p; \qquad x \longrightarrow \frac{1}{\alpha}x$.

(c) Show that any quadratic Hamiltonian of the form $H = \frac{p^2}{2m} + \frac{1}{2}m^2\omega x^2 + 2Fxp$ with $F^2 < \frac{m\omega}{4}$ can be brought into the canonical form

$$U^\dagger H U = \frac{p^2}{2m} + \frac{1}{2}m^2\Omega x^2$$

by a transformation of the type

$$U = e^{i\alpha A}e^{i\beta B}.$$

Find the parameteres of the transformation α and β and the new frequency Ω.

Hint: Find the rotation matrix that diagonalizes the quadratic form which defines the Hamiltonian H. Thus find the rotation angle. Then apply the scaling transformation to get the coefficient of p^2 to the desired form.

B5. Consider a particle in a symmetric potential shown in the Figure below. The walls at x=0 and x=2a+b are infinite. Find approximate expressions of the energy levels and the wave functions of the particle if $E \ll V_0$; the penetrability of the barrier is small ($2mV_0b^2/h^2 \gg 1$) but not zero!

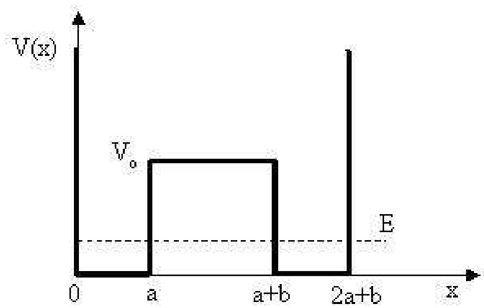

Figure 1: For problem **B5**.

B6. An unpolarized beam of atoms with spin quantum number 1/2 and zero orbital angular momentum passes through a Stern-Gerlach magnet whose magnetic field is along the z axis.

(a) What would you detect on the other side of the Stern-Gerlach magnet?

(b) Suppose the initial beam is polarized along the x axis. What will be the outcome of the experiment now?

(c) Now assume that the initial beam is polarized along direction A at an angle θ to the z axis. What will be the ratio of the number of atoms with spins parallel vs. anti-parallel to the z axis at the output of the Stern-Gerlach apparatus?

(d) Describe how would your answer in (a) be different if the atoms with spin quantum number 1/2 had non-zero orbital angular momentum.