Preliminary Exam : Quantum Physics

August 22, 2003, 9:00 a.m. - 1:00 p.m.

Please answer **3 QUESTIONS** from each of the two sections.

Please use a separate book **FOR EACH QUESTION.**

Some of the following information may be useful.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\int_0^\infty dx\; e^{-a^2x^2} = \frac{\pi^{1/2}}{2a} \quad , \quad \int_0^\infty dx\; xe^{-a^2x^2} = \frac{1}{2a^2}$$

$$\text{Hermite polynomial}: H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

$$\text{associated Laguerre}: L_{n+l}^{2l+1}(r) = \sum_{k=0}^{n-l-1}(-1)^{k+2l+1}\frac{[(n+l)!]^2 r^k}{(n-l-1-k)!(2l+1+k)!k!}$$

$$\text{Legendre polynomial}: P_l(w) = \frac{1}{2^l l!}\frac{d^l}{dw^l}(w^2-1)^l$$

$$\text{associated Legendre polynomial}: P_l^m(w) = (1-w^2)^{|m|/2}\frac{d^{|m|}}{dw^{|m|}}P_l(w)$$

$$\text{spherical harmonic}: Y_l^m(\theta,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

$$\text{spherical Bessels}: j_l(r) = R_l(r)\frac{\sin r}{r} + S_l(r)\frac{\cos r}{r} \quad , \quad n_l(r) = R_l(r)\frac{\cos r}{r} - S_l(r)\frac{\sin r}{r},$$

$$\text{with} \quad R_l(r) + iS_l(r) = \sum_{s=0}^{l}\frac{i^{s-l}(l+s)!}{2^s s!(l-s)!}r^{-s},$$

$$\text{and with asymptotic behavior}: \quad j_\ell(r) \to \sin(r-\ell\pi/2)/r \quad , \quad sn_\ell(r) \to \cos(r-\ell\pi/2)/r.$$

Section I: Statistical Mechanics

1. Consider an infinite chain of identical masses m constrained to move along a fixed linear axis. Each mass is connected to its two nearest neighbors by identical massless springs with spring constant $m\omega^2$ and equilibrium length a. The entire system is in thermal equilibrium at temperature T where $kT \ll m\omega^2 a^2$ but $kT \gg \hbar\omega$. What are the r.m.s. fluctuations in the coordinate $(x_i - x_{i+n})$ for two masses separated by n links along an axis?

2. You are given four particles which have altogether four different states available to them (i.e. each particle could occupy any of the four states).

   How many distinct configurations are allowed for the system if :

   (i) the four particles are totally distinguishable,

   (ii) the four particles are identical bosons,

   (iii) two of the particles are identical bosons and two are some other bosons which are identical to each other but not to the first two bosons,

   (iv) the four particles are identical fermions,

   (v) two of the particles are identical fermions and two are some other fermions which are identical to each other but not to the first two fermions.

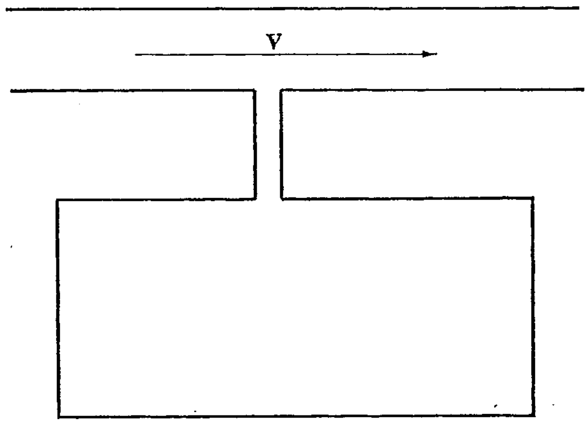
3. Consider a molecular dynamics (MD) simulation of N classical particles under conditions of constant volume V, and temperature T. During the run, the internal part of the virial $\mathcal{V}_{\text{int}}$ is calculated at each integration step and stored to a file:

$$\mathcal{V}_{\text{int}} = \sum_{i=1}^{N} \sum_{j>i}^{N} \mathbf{f}_{ij} \cdot \mathbf{r}_{ij}$$

   where $\mathbf{f}_{ij}$ is the force excerted by particle i onto particle j, and $\mathbf{r}_{ij}$ denotes the inter-particle vector.

   (a) Show how the pressure of the system can be obtained from the sequence $\mathcal{V}_{\text{int}}(t)$.

   (b) By considering $\frac{\partial}{\partial\beta}\langle\mathcal{V}_{\text{int}}\rangle$, where $\beta = 1/kT$, obtain the thermal pressure coefficient $\gamma_T = \left(\frac{\partial P}{\partial T}\right)_V$ from the covariance $\langle\Delta\mathcal{V}_{\text{int}}\Delta E\rangle$ of the internal virial and the total energy .



4. Consider a classical ideal gas streaming uniformly in a tube, connected to a gas reservoir through a thin capillary as shown in the illustration. The reservoir contains the same gas at rest and is at the same temperature as the streaming gas.

(a) Using the fact that the total momentum **P** in the streaming gas is conserved, show that its canonical partition function is given by

$$Q_N(V,T,\mathbf{P}) = \frac{V^N}{N!}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3N}{2}} e^{\frac{Nm\mathbf{v}^2}{2kT}},$$

where **v** is the streaming velocity. (Hint: Use the method of Lagrange multipliers to ensure the constraint **P** = const., and demonstrate that the corresponding Lagrange multiplier is proportional to the streaming velocity **v**.)

(b) What is the chemical potential of the streaming gas?

(c) Equate the chemical potentials of the streaming and resting gases to determine the ratio n_1/n_2 of the number density of the gas in the stream (n_1) to the number density of the gas in the reservoir (n_2).

Section II: Quantum Mechanics

5. A one-dimensional quantum-mechanical harmonic oscillator has a Hamiltonian $H = p^2/2m + m\omega^2 q^2/2$.

   (a) In the Heisenberg picture derive the equations of motion of the quantized position and momentum operators $q(t)$ and $p(t)$.

   (b) In this same Heisenberg picture calculate the following 3 unequal time commutators:

   $$[q(t), q(t')] \quad , \quad [q(t), p(t')] \quad , \quad [p(t), p(t')].$$

   (c) A convenient rewriting of the harmonic oscillator Hamiltonian may be obtained by setting

   $$q = (\hbar/2m\omega)^{1/2}(a + a^\dagger) \quad , \quad p = i(\hbar m\omega/2)^{1/2}(a^\dagger - a)$$

   The state $|0\rangle$ is defined by the condition $a|0\rangle = 0$. Show that $|0\rangle$ is the ground state of the harmonic oscillator, and use this information to construct the q dependence of the ground state wave function.

6. A one-dimensional quantum-mechanical harmonic oscillator with Hamiltonian $H = p^2/2m + m\omega^2 q^2/2$ is in its second excited state $|2\rangle$. It is subjected to a perturbatuion

   $$V(t \geq 0) = \alpha x^2 \exp^{-t/\tau}$$

   where τ and α are positive constants.

   (a) To what states can it make transitions in first order perturbation theory?

   (b) Calculate the corresponding transition probabilities to these states after the perturbation has been applied for a long time ($t \to \infty$).

7. (a) Consider a particle in a one-dimensional finite depth square-well potential of the form

   $$V(x) = -V_0\, \Theta(a - |x|)$$

   where $2a$ is the width of the well that determines the short-distance behavior of $V(x)$, and Θ is the normalized step function $\Theta(u) = \begin{cases} 0, & u < 0 \\ 1, & u > 0 \end{cases}$.

   What restrictions on V_0 are implied by the condition that the system possess exactly one bound state?

   (b) Consider a particle in a three-dimensional radial finite depth square-well potential of the form

   $$V(r) = -V_0\, \Theta(a - r)$$

   where a is the radius of the well that determines the short-distance behavior of $V(r)$. What are the allowed values of V_0 for which the system has one and only one s-wave bound state?

8. Consider a quantum system composed of a free rigid rod with a uniform charge and mass density along its length. The rod is of length a, mass m and total charge q.

   (a) What is the Hamiltonian for the free system?

   (b) What are the eigenstates and eigenvalues of the system when the center of the rod is held fixed?

   (c) What is the magnetic moment of the rod in these eigenstates?