Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

You are allowed to use a result stated in one part of a problem in the subsequent parts even if you cannot derive it. On the last page you will find some potentially useful formulas.
Problem 1. Let \( H = H_{\text{kin}} + V(\vec{x}) \) be a single-particle Hamilton operator with \( H_{\text{kin}} = \frac{\vec{p}^2}{2M} \) and \( M \) the mass of the particle. Consider the operator \( D = \frac{1}{2}(\vec{x} \cdot \vec{p} + \vec{p} \cdot \vec{x}) \).

(a) Calculate the commutators \([D, \vec{x}]\) and \([D, \vec{p}]\).

(b) Calculate \([D, F(\vec{x})]\) and \([D, G(\vec{p})]\) where \( F \) and \( G \) are differentiable functions. You may want to work out the commutators in position or momentum space.

(c) Let \( H\ket{E_i} = E_i\ket{E_i} \). Calculate \([D, H]\) and prove that \(2 \langle E_i | H_{\text{kin}} | E_i \rangle = \langle E_i | \vec{x} \cdot \vec{\nabla} V(\vec{x}) | E_i \rangle\), which is the quantum mechanical virial theorem.

Problem 2. Let us study the two-dimensional isotropic oscillator with the potential \( U(x, y) = \frac{1}{2} m\omega^2(x^2 + y^2) \).

(a) Find the energy levels of the system with the quantum numbers of the \( x \) and \( y \) oscillators equal to \( n_x \) and \( n_y \). What are the degeneracies of the energy levels?

(b) Show that the Hamiltonian of the system commutes with (what would be the \( z \) component of the three-dimensional) angular momentum

\[
L = \frac{\hbar}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).
\]

From the results of parts (a) and (b) we conclude that one can diagonalize both the Hamiltonian and the angular momentum operator with eigenfunctions \( \psi_{n\ell} \), where \( n = 0, 1, \ldots \) labels the solutions of the radial Schrödinger equation so that \( E_{n\ell} = \hbar\omega(n + 1) \), and \( \ell \) labels the angular states so that \( L\psi_{n\ell} = \hbar\ell\psi_{n\ell} \). It turns out that for a given \( n \), the value of \( \ell \) runs from \(-n\) to \( n\) in steps of two.

(c) For the three lowest-energy levels \( n = 0, 1, 2 \) write explicitly the eigenfunctions \( \psi_{n\ell}(x, y) \), which simultaneously diagonalize \( H \) and \( L \), in terms of the eigenfunctions of the one-dimensional harmonic oscillator states \( \psi_{n_x}(x) \) and \( \psi_{n_y}(y) \).

Recall that the unit-normalized lowest wave functions of the harmonic oscillator are of the form

\[
\psi_0(x) = Ce^{-\frac{\sigma^2 x^2}{2}}, \quad \psi_1(x) = C\sqrt{2}\sigma x e^{-\frac{\sigma^2 x^2}{2}}, \quad \psi_2(x) = C\sqrt{2}(\sigma^2 x^2 - \frac{1}{2}) e^{-\frac{\sigma^2 x^2}{2}}
\]

with \( \sigma = \sqrt{m\omega/\hbar} \) and \( C \) being an irrelevant common normalization constant.
**Problem 3.** Consider a two-level system with the Hamiltonian

\[
\frac{H}{\hbar} = \lambda (|1\rangle \langle 2| + |2\rangle \langle 1|),
\]

where \( \lambda \) is real and \{\( |1\rangle, |2\rangle \)\} makes an orthonormal basis in the corresponding Hilbert space.

(a) The system starts at \( t = 0 \) in the state \( |\psi\rangle = C_1 |1\rangle + C_2 |2\rangle \). Show that at time \( t \) the state is

\[
|\psi(t)\rangle = C_1 (\cos \lambda t |1\rangle - i \sin \lambda t |2\rangle) + C_2 (\cos \lambda t |2\rangle - i \sin \lambda t |1\rangle).
\]

(b) In particular, suppose the system starts in state \( |1\rangle \). What is the probability that it would be found in state \( |2\rangle \) if a measurement were made at time \( t \)?

(c) Continuing from part (b), suppose that measurements are actually made at two times \( t_1 \) and \( t_2 \), with \( t_2 > t_1 \). What is the probability that the system is found in state \( |2\rangle \) at both times?

**Problem 4.**

(a) Show that if for some linear operator \( Q \) the inner product equality \( \langle \psi, Q\phi \rangle = 0 \) is valid for all vectors \( \psi \) and \( \phi \), then \( Q \) is the zero operator that maps all vectors to the zero vector.

(b) Suppose every expectation value of an operator \( Q \) is zero, \( \langle \psi, Q\psi \rangle = 0 \) for all vectors \( \psi \). Of course, then we also have \( \langle \psi + \lambda \phi, Q(\psi + \lambda \phi) \rangle = 0 \), no matter what the vectors \( \psi \) and \( \phi \) and the scalar \( \lambda \) are. By applying this observation with \( \lambda = 1 \) and \( \lambda = i \), show that \( \langle \psi, Q\phi \rangle = 0 \) for all \( \psi \) and \( \phi \). By combining with the result of part (a), we have the result \( \langle \psi, Q\psi \rangle = 0 \forall \psi \Rightarrow Q = 0 \).

(c) Every operator \( Q \) may be decomposed trivially in the form \( Q = Q_1 + iQ_2 \) with \( Q_1 = \frac{1}{2}(Q + Q^\dagger) \) and \( Q_2 = -\frac{i}{2}(Q - Q^\dagger) \). Show that \( Q_1 \) and \( Q_2 \) are hermitian.

(d) The expectation value of a hermitian operator in every state is real. Show that the reverse also holds true: An operator such that its expectation value in every state is real is necessarily hermitian. Combining the results of parts (b) and (c) is one possible way to proceed.
Problem 5. Suppose we know the normalized eigenstates $|n^{(0)}\rangle$ and the corresponding non-degenerate energies $E_n^{(0)}$ of the unperturbed Hamiltonian $H^{(0)}$. Let $V$ be a “small” time-independent perturbation.

(a) It is possible to use perturbation theory to write the eigenstates $|n\rangle$ and the corresponding energies of the Hamiltonian $H = H^{(0)} + V$ as series of the form $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \ldots$ and $E_n = E_n^{(0)} + E_n^{(1)} + \ldots$, where the quantities with the superscript $(j)$ scale with the $j$th power of the strength of the perturbation $V$. Show that the first- and second-order corrections to the energies are

$$E_n^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle, \quad E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)} | V | m^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}.$$

(b) Consider now the one-dimensional Hamilton operator $H^{(0)} = \frac{p^2}{2m} + \frac{1}{2}m^2 \omega^2 x^2$ and the perturbation $V = -Fx$. Using the formalism developed in part (a), calculate the eigenenergies of the Hamiltonian $H = H^{(0)} + V$ in first and second order in perturbation theory.

(c) Find the exact eigenenergies of the Hamiltonian $H = H^{(0)} + V$ of part (b), and check your perturbative results against them.
\[
\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty
\]

\[
\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \text{Re}(\alpha) > 0
\]

\[
\int_{0}^{\infty} dx \, x \exp(-\alpha x^2) = \frac{1}{2\alpha}
\]

\[
a = \sqrt{\frac{m \omega}{2\hbar}} \left(x + i \frac{p}{m \omega}\right)
\]