QUANTUM MECHANICS/ STATISTICAL PHYSICS

Preliminary Examination

January 16, 2009

9:00 - 15:00 in P-121

Answer a total of **SIX** questions, choosing **TWO** from Section A, and **FOUR** from Section B. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

Stirling's approximation for large N is $\mathrm{N!} \simeq \mathrm{N}\ln(\mathrm{N}) - \mathrm{N}$

$\int_0^\infty e^{-ax^2} dx = \sqrt{\pi/4a}$

$\int_0^\infty x\, e^{-ax^2} dx = 1/(2a)$

Note: $\int_0^\infty x^n e^{-ax^2} dx$ for any integer $n > 0$ can be found by taking suitable derivatives of the above two integrals with respect to a.

SECTION A - STATISTICAL PHYSICS

A1. Consider a system consisting of $N \gg 1$ non-interacting distinguishable particles. Each particle has two possible energy levels: ϵ_0 and $-\epsilon_0$.

(a) Calculate the statistical weight W_M of a state with total energy $E = -M\epsilon_0$. Assume M is an integer such that $0 \leq M < N$ while N-M is even and $\gg 1$.

(b) What is the entropy of this state?

(c) How does the energy of the system vary with temperature T?

(d) Calculate the specific heat of this system.

A2. A degenerate gas of (T=0) fermions with spin quantum number s=5/2 occupies an infinitely large (three-dimensional) volume. Dependence of the particle energy $\epsilon(\vec{\mathbf{p}})$ on the momentum $\vec{\mathbf{p}}$ is given by the relativistic formula

$$\epsilon(\vec{\mathbf{p}}) = \sqrt{m^2c^4 + p^2c^2},$$

where m is the particle mass and c is the speed of light.

(a) Calculate the number density n of particles and their energy density w if the Fermi momentum is p_0.

(b) Find the dependence of the gas pressure P on the Fermi momentum p_0 for the asymptotic case $mc/p_0 \to 0$.

(c) Find how n and w depend on p_0 if this gas occupies a two-dimensional space.

A3. In a simple (intrinsic) semiconductor, valence states are fully occupied while the conduction states are completely empty at zero temperature. Valence and conduction bands are separated by an energy gap $E_g = E_c - E_v > 0$ where E_c and E_v are the bottom and the top of the conduction and the valence bands respectively. As temperature increases, it is possible to excite the valence electrons over to the conduction band creating holes in the valence band. In this model, take the energy $\epsilon(\vec{\mathbf{k}})$ of an electron in the conduction band to be

$$\epsilon(\vec{\mathbf{k}}) = E_c + \hbar^2 k^2/2m_e$$

and that of a hole in the valence band to be

$$\epsilon(\vec{\mathbf{k}}) = E_v - \hbar^2 k^2/2m_h$$

where m_e and m_h correspond to effective masses of the electron and hole.

(a) Find the densities of states $\mathcal{D}_e(\epsilon)$ and $\mathcal{D}_h(\epsilon)$ of electrons and holes where ϵ identifies the single particle energy of electrons in the conduction band or that of holes in the valence band in 3-dimensions, confined to a cubic box of side $L \to \infty$.

(b) Write down integral expressions for the number density of electrons n_e that can be excited to the conduction band and the number density of holes n_h left behind in the valence band at temperature T. (Hint: Use the densities of states calculated above in your expressions.)

(c) Simplify your expressions assuming that $|\epsilon - \mu|/k_B T \gg 1$ where ϵ identifies the single electron or hole energy and μ is the chemical potential. Show that the product of electron and hole densities, $n_e n_h$, is independent of the chemical potential at a given temperature. (This result has important implications for carrier concentration in an intrinsic semiconductor.)

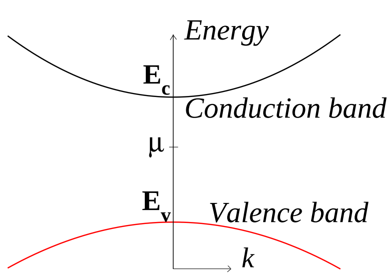

Figure 1: For problem **A3**.

SECTION B - QUANTUM MECHANICS

B1. A quantum system is described by the eigenstate with an orbital momentum l and its projection m on the axis of quantization z. Calculate the average value $\left\langle \hat{L}_\xi \right\rangle$ of the projection of the angular momentum on the $\vec{\xi}$-axis, the average value of $\left\langle \hat{L}_\xi^2 \right\rangle$ and the average value of quadratic fluctuations of this projection $\left\langle (\hat{L}_\xi - \left\langle \hat{L}_\xi \right\rangle)^2 \right\rangle$. Here, the $\vec{\xi}$-axis is defined by the polar angle α and azimuthal angle ϕ in the coordinate frame of quantization.

B2. (a) In a system of two identical particles, explain why simultaneous eigenstates of the operators $\vec{\mathbf{S}}^2$ and S_z should exist. Here $\vec{\mathbf{S}} = \vec{\mathbf{S}}_1 + \vec{\mathbf{S}}_2$ is the total spin operator of the system and S_z, its z-component.

(b) Write down such eigenstates in terms of one-particle states for a system of two spin 1/2 particles. Discuss the symmetry of these states and find the corresponding eigenvalues of the two operators $\vec{\mathbf{S}}^2$ and S_z.

(c) If the above system consists of two identical spin 1 particles, write down the corresponding eigenvalues of the simultaneous eigenstates of the (same) two operators. Find any possible antisymmetric and simultaneous eigenstates in this case.

B3. A quantum particle of mass m is initially in the ground state of a Dirac potential well $V(x) = -\alpha\ \delta(x)$ where α is a positive constant and $\delta(x)$ is the Dirac delta function. Suddenly, at time $t = 0$, the potential starts moving along the x-axis with speed u. Calculate the probability that the particle is bound by the moving potential well.

Hint: The initial wavefunction should be written in the moving coordinate frame. Use a Galilean transformation of this wavefunction.

B4. Consider two non-identical, non-interacting particles of mass M that are constrained to move on a circle of radius R.

(a) Write down the Schrödinger equation for this problem and find the eigenfunctions and energy levels of this system.

(b) Specify the degeneracy of the first 3 energy levels.

(c) Now allow the particles to have a weak attractive interaction given by the following perturbation to the Hamiltonian:

$$V(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = (k/2)|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|^2$$

with $k < 0$ being a constant. Here $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ are the positions of the two particles. Calculate, to lowest order, the shift in energy of the ground state of the system.

B5. Consider a singly ionized $^6He^+$ atom. The nucleus can undergo beta decay, transforming the ion into doubly ionized $^7Li^{++}$. In the following, take the nuclear mass to be infinitely large relative to the electron mass.

(a) Write down the ground state energy and wavefunction of $^6He^+$. Verify that your wavefunction is properly normalized.

(b) Starting with the $^6He^+$ in its ground state, and assuming the beta decay process to be instantaneous, calculate the probability that the resulting $^7Li^{++}$ is in its ground state. Neglect any interaction of the atomic electron with the emitted beta particle.

B6. Consider a 3-state system described by the Hamiltonian

$$H = \lambda \left|Q\right\rangle \left\langle Q\right| + \mu\{\left|L\right\rangle \left\langle R\right| + \left|R\right\rangle \left\langle L\right|\}$$

where $|Q\rangle$, $|L\rangle$ and $|R\rangle$ represent position eigenkets of an electron in 3 different orthonormal states.

(a) Find the eigenkets and eigenvalues of the Hamiltonian H defined above, assuming that λ and μ are real and have units of energy.

(b) Write down the time evolved state $|\psi(t = T)\rangle$ provided that the initial state is given by $|\psi(t = 0)\rangle = |R\rangle$. Find the probabilities that the particle can be found in the states (i) $|Q\rangle$, (ii) $|L\rangle$ after time T.