1. A static charge distribution produces a radial electric field of the form: \( E(r) = \frac{A \exp(-br)}{r} \). The direction of \( \mathbf{E} \) is along the vector \( \mathbf{r} \), and \( A \) and \( b \) are constants.

   (a) Determine the dependence of the charge density \( \rho(r) \) on \( r \) and draw a sketch of it.

   (b) What is the total charge \( Q \)?

2. (a) Find the work required to bring a charge \( q \) from infinity to a distance \( d \) away from a semi-infinite dielectric medium (see figure) whose dielectric constant is \( K \).

   (b) Calculate the work required if instead the plane is perfectly conducting and grounded.
3. (a) Consider a circular loop with radius $R$ carrying a current $I$ flowing counter clockwise in the $x$-$y$ plane around the origin, see Fig. (a). Determine the magnetic induction along the $z$-axis as a function of $z$, i.e. $B_z(x)|_{x=(0,0,z)} \equiv B(z)$.

(b) Consider now two current loops, as shown in Fig. (b), where the distance between the centers of the two loops is $2L$. Determine the magnetic induction along the $z$-axis.

(c) Expand the result from part (b) to appropriate order in $z$ to determine for which $L$ (expressed in terms of $R$) the magnetic induction along the $z$-axis is $B(z) = B_0 + O(z^4)$ for $z \ll L$. Determine $B_0$ as function of $I$ and $R$.

4. (a) Consider electromagnetic potentials $\phi(x,t)$ and $A(x,t)$ such that the homogeneous Maxwell equations are automatically satisfied. Show that the inhomogeneous Maxwell equations in the presence of a charge density $\rho(x,t)$ and a current density $j(x,t)$ decouple as

$$\nabla^2 \phi(x,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(x,t) = -\frac{1}{\varepsilon_0} \rho(x,t), \quad \nabla^2 A(x,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A(x,t) = -\mu_0 j(x,t)$$

if one imposes the Lorenz gauge condition $\nabla \cdot A(x,t) + \frac{1}{c^2} \frac{\partial}{\partial t} \phi(x,t) = 0$ on the potentials. (There are no medium effects in this problem, i.e. $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\varepsilon_0 \mu_0 c^2 = 1$.)

(b) Set $A(x,t) = -\frac{\varepsilon_0}{\varepsilon} E_0 \frac{\omega}{c} \sin(\omega t) \mathbf{e}_z$ in the Lorenz gauge. Determine $\phi(x,t)$, $\rho(x,t)$, $j(x,t)$ and check explicitly whether electric charge is conserved. Determine also $E(x,t)$, $B(x,t)$ and investigate whether energy is conserved or whether this is an open system.
Vector Formulas

\[ a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \]
\[ a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \]
\[ (a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \]
\[ \nabla \times \nabla \psi = 0 \]
\[ \nabla \cdot (\nabla \times a) = 0 \]
\[ \nabla \times (\nabla \times a) = \nabla(\nabla \cdot a) - \nabla^2 a \]
\[ \nabla \cdot (\psi a) = a \cdot \nabla \psi + \psi \nabla \cdot a \]
\[ \nabla \times (\psi a) = \nabla \psi \times a + \psi \nabla \times a \]
\[ \nabla (a \cdot b) = (a \cdot \nabla)b + (b \cdot \nabla)a + a \times (\nabla \times b) + b \times (\nabla \times a) \]
\[ \nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b) \]
\[ \nabla \times (a \times b) = a(\nabla \cdot b) - b(\nabla \cdot a) + (b \cdot \nabla)a - (a \cdot \nabla)b \]

If \( x \) is the coordinate of a point with respect to some origin, with magnitude \( r = |x|, \) \( n = x/r \) is a unit radial vector, and \( f(r) \) is a well-behaved function of \( r, \) then

\[ \nabla \cdot x = 3 \]
\[ \nabla \times x = 0 \]
\[ \nabla \cdot [nf(r)] = \frac{2}{r} f + \frac{\partial f}{\partial r} \]
\[ \nabla \times [nf(r)] = 0 \]
\[ (a \cdot \nabla)nf(r) = \frac{f(r)}{r} \left[ a - n(a \cdot n) \right] + n(a \cdot n) \frac{\partial f}{\partial r} \]
\[ \nabla (x \cdot a) = a + x(\nabla \cdot a) + i(L \times a) \]

where \( L = \frac{1}{i} (x \times \nabla) \) is the angular-momentum operator.
Theorems from Vector Calculus

In the following $\phi$, $\psi$, and $A$ are well-behaved scalar or vector functions, $V$ is a three-dimensional volume with volume element $d^3x$, $S$ is a closed two-dimensional surface bounding $V$, with area element $da$ and unit outward normal $n$ at $da$.

\[
\int_V \nabla \cdot A \, d^3x = \int_S A \cdot n \, da \quad \text{(Divergence theorem)}
\]

\[
\int_V \nabla \psi \, d^3x = \int_S \psi n \, da
\]

\[
\int_V \nabla \times A \, d^3x = \int_S n \times A \, da
\]

\[
\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, d^3x = \int_S \phi n \cdot \nabla \psi \, da \quad \text{(Green's first identity)}
\]

\[
\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot n \, da \quad \text{(Green's theorem)}
\]

In the following $S$ is an open surface and $C$ is the contour bounding it, with line element $dl$. The normal $n$ to $S$ is defined by the right-hand-screw rule in relation to the sense of the line integral around $C$.

\[
\int_S (\nabla \times A) \cdot n \, da = \oint_C A \cdot dl \quad \text{(Stokes's theorem)}
\]

\[
\int_S n \times \nabla \psi \, da = \oint_C \psi \, dl
\]
Explicit Forms of Vector Operations

Let \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and \( A_1, A_2, A_3 \) be the corresponding components of \( \mathbf{A} \). Then

\[
\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial \psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial \psi}{\partial x_3}
\]

\[
\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}
\]

\[
\nabla \times \mathbf{A} = \mathbf{e}_1 \left( \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left( \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)
\]

\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}
\]

Cylindrical

\( (\rho, \phi, z) \)

\[
\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial \rho} + \mathbf{e}_2 \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} + \mathbf{e}_3 \frac{\partial \psi}{\partial z}
\]

\[
\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho A_1 \right) + \frac{1}{\rho} \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}
\]

\[
\nabla \times \mathbf{A} = \mathbf{e}_1 \left( \frac{1}{\rho} \frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho} \right) + \mathbf{e}_3 \frac{1}{\rho} \left( \frac{\partial \rho}{\partial \phi} \left( \rho A_2 \right) - \frac{\partial A_1}{\partial \phi} \right)
\]

\[
\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}
\]

Spherical

\( (r, \theta, \phi) \)

\[
\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial r} + \mathbf{e}_2 \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \mathbf{e}_3 \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi}
\]

\[
\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A_1 \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta A_2 \right) + \frac{1}{r \sin \theta} \frac{\partial A_3}{\partial \phi}
\]

\[
\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta A_3 \right) - \frac{\partial A_2}{\partial \phi} \right] + \mathbf{e}_2 \left[ \frac{1}{r \sin \theta} \frac{\partial A_1}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( r A_3 \right) \right] + \mathbf{e}_3 \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r A_2 \right) - \frac{\partial A_1}{\partial \theta} \right]
\]

\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\]

\[
\left[ \text{Note that } \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (r \psi) \right]
\]