Preliminary Exam : Classical Physics

August 21, 2003, 9:00 a.m. - 1:00 p.m.

Please answer **3 QUESTIONS** from each of the two sections.

Please use a separate book **FOR EACH QUESTION.**

Section I: Classical Mechanics

1. An object of mass m is initially at rest on a horizontal frictionless table. The mass is attached to a spring whose spring constant is k; the other end of the spring is attached to a wall. Take the origin to be the initial position of the mass and assume that there is no damping. Starting at $t = 0$, the mass is subjected to an applied force given by $F_0 \sin \omega t$.

   (a) Find the motion $x(t)$.

   (b) Describe the behavior of the amplitude of $x(t)$ as a function of the frequency ω.

2. Consider the Atwood Engine, as shown in Figure 1, which consists of two masses, m_1 and m_2 hanging vertically, suspended over a frictionless, massless pulley of radius r by a thin rope of total length L.

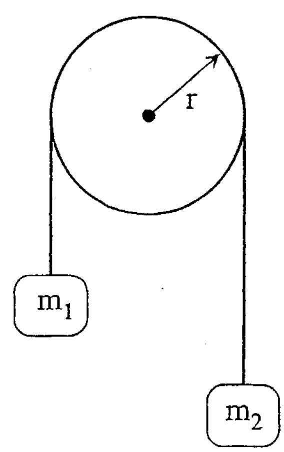

Figure 1:

   (a) Use the Lagrangian method to solve for the motion of the mass m_1.

   (b) Use the Hamilton-Jacobi equation to solve for the motion of the mass m_1 .

3. A disk of mass M_1, radius R and moment of inertia $I = \frac{1}{2}M_1R^2$ about its center, rolls without slipping along a horizontal shelf, as shown in Figure 2. Hanging from its center of rotation is another mass, M_2. The string has a length L.

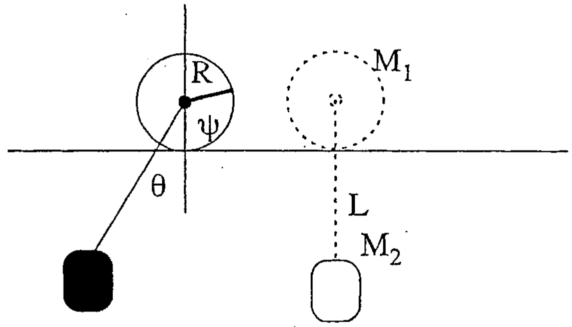

Figure 2:

   (a) Write down the exact Lagrangian in terms of the variables θ and ψ.

   (b) Simplify the Lagrangian for small oscillations. (You only need to keep terms in the kinetic energy up to order θ, while for the potential energy you must keep terms to order θ^2). Find the equations of motion for θ and ψ that follow from this simplified Lagrangian.

   (c) Find the normal modes and calculate the frequencies of small oscillations from the Lagrangian derived in part (b).

4. Consider the gravitational attraction of two masses to one another, with one mass being much heavier than the other.

   (a) Write down the appropriate Lagrangian and identify a cyclic coordinate and its corresponding conserved quantity.

   (b) Eliminate the time dependence in order to derive an equation for the shape of the orbit. Solve this orbit equation using the integral (**hint:** define $u = \frac{1}{r}$) :

$$\int \frac{du}{\sqrt{\alpha + \beta u - u^2}} = -\arccos\left[\frac{2u - \beta}{\sqrt{\beta^2 + 4\alpha}}\right]$$

   thereby showing that the orbit has the form of a conic section, with one focus at the origin, the formula for which is (here, e is the "eccentricity" of the orbit):

$$\frac{1}{r} = C\,(1 + e\,\cos(\theta - \theta_0))$$

   (c) Sketch the effective potential, and list the possible forms for the shape of the orbit.

Section II: Electromagnetism

5. A point charge q is situated a distance d from a grounded conducting plane of infinite extent. Calculate the induced surface charge density on the plane, and show by direct integration of this charge density that the total induced charge on the plane is in fact $-q$.

6. Consider a very long cylinder of radius R which is made out of a magnetic material characterized by a permeability μ. The cylinder is placed in a region of free space containing an (initially) uniform magnetic field $\vec{B}_0$, which is perpendicular to the axis of the cylinder. Using the magnetic scalar potential ϕ^*, where $\vec{H} = -\vec{\nabla}\phi^*$, and Laplace's equation $\nabla^2\phi^* = 0$, solve for the magnetic field $\vec{B}$ inside the cylinder.

   Note: associated with Laplace's equation are the so-called cylindrical harmonics: 1, $\ln r$, $r^n \cos\theta$, $r^{-n}\cos n\theta$, $r^n \sin\theta$, $r^{-n}\sin n\theta$.

7. Consider a large wire loop (loop 1) of radius R in the $x-y$ plane, and a very small wire loop (loop 2) of radius a parallel to the $x-y$ plane at a position $z = d$, as shown in Figure 3. You can assume that the small loop is described just by its dipole moment. Note that the magnetic field for a magnetic dipole oriented in the z-direction is

$$B_r = \frac{2\mu_0 m}{4\pi r^3}\cos\theta \quad , \quad B_\theta = \frac{\mu_0 m}{4\pi r^3}\sin\theta$$

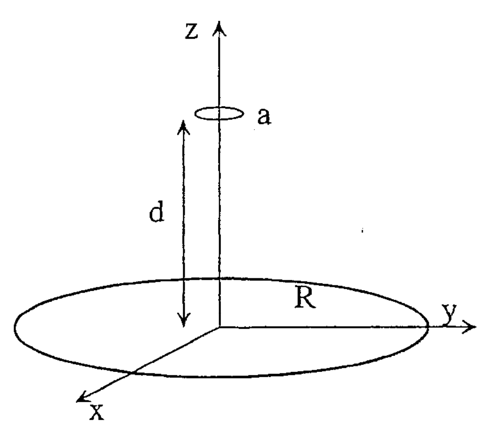

Figure 3:

   (a) Calculate the mutual inductances, M_{12} and M_{21}, between the loops, and show that they are the same.

   (b) Now consider the case $d = 0$. Set up the integral for M_{21} in a way that can be evaluated, and obtain the appropriate limit of your answer in part (a). (Hint: consider the magnetic flux through loop 1 due to a current in loop 2.)

   (c) If the small loop carries a current I and is moving at a constant velocity along the z-axis, at what value of z is the induced EMF in the large loop the greatest?

8. (a) Write down Maxwell's equations for the propagation of an electromagnetic wave of frequency ω along a long straight waveguide which has perfectly conducting walls and whose interior is a dielectric of permeability μ and permittivity ϵ.

(b) Consider the propagation of a TEM mode (*i.e.*, one for which there is no electric or magnetic field in the direction of propagation along the waveguide). Show that the propagation of such a mode is impossible if the waveguide is hollow, but is possible for a coaxial waveguide.

(c) Find the form of the electric and magnetic fields for a TEM mode inside a long straight cylindrical coaxial waveguide with circular cross-sections.

note: the Laplacian in cylindrical coordinates is $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$