Classical Mechanics / Electricity and Magnetism

General Exam Questions for January, 2005

Instructions

Answer three questions from each of the two sections, for a total of six problems. Put each of your solutions in a separate answer book. Make sure that you label and sign your name on the cover of each book.

I. Classical Mechanics

1. A classical particle of mass m is moving in a velocity-dependent potential, with a Lagrangian given by

$$L = \frac{1}{2}m\dot{x}^2 + a\vec{v}\cdot\vec{A}$$

where $\vec{A} = \vec{\nabla}\Phi$. The vector field $\vec{A}(\vec{x})$ does not depend on time.

a) Find the Hamiltonian for this system in canonical variables $\vec{x}$ and $\vec{p}$.

b) Solve for the motion of the particle $\vec{x}(t)$, for general initial conditions. Show that the result is the same as for a free particle of mass m.

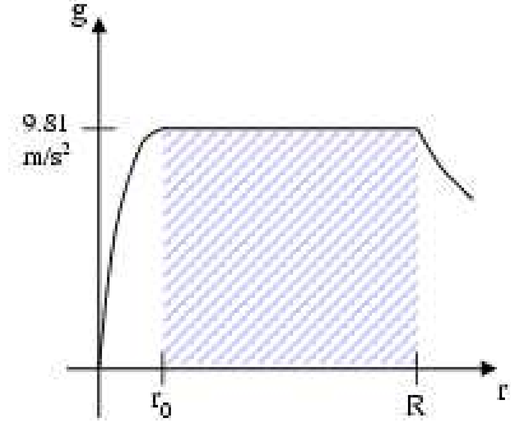
c) Demonstrate a canonical transformation from $\vec{x}, \vec{p}$ to new canonical variables $\vec{x}\,', \vec{p}\,'$ such that the transformed Hamiltonian is the free one $H' = \frac{1}{2m}p'^2$.

2. Two thin homogeneous rods, each of mass m and length ℓ, hang from a horizontal ceiling under gravity with acceleration g. The rods are connected by a massless spring with spring constant k, in such a way that the spring is unstretched when the rods hang straight down.

   a) Analyze the small-oscillation modes assuming that the rods only move in the plane defined by their equilibrium positions.

   b) Without doing any major calculations, describe the normal modes of small oscillations that are not confined to said plane.

3. Consider a model of the earth consisting of a sphere of radius R that is composed of a single material whose density depends on pressure. Consider a borehole drilled in the earth that extends from the surface at $r = R$ down to a $r = r_0$ near the center of the planet. Suppose that the local gravitational constant g is a constant, independent of r for $r_0 < r < R$, as shown in the figure.



   a) What is the density $\rho(r)$ as a function of r for $r_0 < r < R_0$?

   b) What is the local velocity of sound in the planetary interior as a function of depth for $r_0 < r < R$?

   Hint: The velocity of sound in a bulk medium is given by $\sqrt{\kappa/\rho}$, where ρ is the density of the medium and κ is the incompressibility defined as $\kappa = -V dP/dV$, with pressure P and volume V.

4. Consider a space shuttle in orbit at a constant 300 km above the surface of the earth. An astronaut inside the shuttle places two identical baseballs, one against the other, so that the line joining their centers passes through the center of the earth. When they are released, both balls and the shuttle are at rest in a co-moving inertial frame. Some time later the balls have drifted apart. Ignoring air currents in the cabin, predict the relative motion of the balls, both direction and magnitude, for times that are much shorter than the orbit period. Note that, since the initial speeds are the same and the distances from the center of the earth are different, at most one of the balls can be in a stable circular orbit.

II. Electricity and Magnetism

1. A very long conducting cylinder of radius a is inserted into a constant (in space and time) electric field $\vec{E}$, so that the direction of the field and the axis of the cylinder are perpendicular to one another. After the transients have died out, what is the new electric field?

   Ignoring boundary conditions, all of the following functions of the usual radial coordinates ρ and ϕ are solutions of the two-dimensional Laplace equation:

$$1\,;\ \ln\rho\,;\ \rho^m\cos m\phi\,,\ \rho^{-m}\cos m\phi\,,\ \rho^m\sin m\phi\,,\ \rho^{-m}\sin m\phi\,,\ m=1,2,\ldots.$$

   The expression for the gradient in cylindrical coordinates is

$$\vec{\nabla} = \hat{e}_\rho\frac{\partial}{\partial\rho} + \hat{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{e}_z\frac{\partial}{\partial z}.$$

2. An electron at position $\vec{x}(t)$ moves in a monochromatic electromagnetic field with

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left(\vec{E}(\vec{r})e^{-i\omega t} + \vec{E}^*(\vec{r})e^{i\omega t}\right).$$

As is appropriate for a non-relativistic electron, we assume that the electron only moves a negligible fraction of a wavelength during one period of the electromagnetic field. Let us also assume that the electromagnetic field has been turned on adiabatically, so that the initial conditions have no influence on the steady-state motion of the particle.

a) Show that the corresponding magnetic field is

$$\vec{B}(\vec{r},t) = \frac{1}{2i\omega}\left(\vec{\nabla}\times\vec{E}(\vec{r})e^{-i\omega t} - \vec{\nabla}\times\vec{E}^*(\vec{r})e^{i\omega t}\right).$$

b) Show that the velocity of the electron is

$$\dot{\vec{x}}(t) = \frac{1}{2}\left(\frac{e\vec{E}(\vec{x})}{im\omega}e^{-i\omega t} - \frac{e\vec{E}^*(\vec{x})}{im\omega}e^{i\omega t}\right).$$

c) Show that the Lorentz force does not average to zero over a period of the electromagnetic field, but the average is

$$\vec{F}(\vec{x}) = -\vec{\nabla}V(\vec{x})\,; \quad V(\vec{x}) = \frac{e^2\left|\vec{E}(\vec{x})\right|^2}{4m\omega^2}.$$

The newly found *ponderomotive potential* $V(\vec{x})$ is one of the key concepts of the physics of an electron in a high-intensity light field.

3. A particle with mass m is moving in a potential V. For an observer in a frame that rotates with angular frequency ω about the z axis, the particle motion is described by the local potential $V = \frac{1}{2}m\omega^2(x^2 + y^2)$. Show that in a non-rotating frame the motion is exactly equivalent to that of a charged particle in a constant magnetic field $\vec{B} = B\hat{e}_z$ such that the magnetic field, charge and mass satisfy the relation $\vec{\omega} = -q\vec{B}/2m$. In this problem, the motion in the z direction separates and can be ignored.

4. Consider a region of space filled with material having a purely ohmic conductivity σ. An unspecified mechanism forces a current $\vec{J} = J\hat{e}_z$ inside a sphere of radius a in this medium. By the continuity equation $\vec{\nabla} \cdot \vec{J} = 0$, the current cannot suddenly drop to zero at the surface of the sphere, so there must be a return current outside of the sphere.

a) In the steady-state, except for possible singularities at the surface of the sphere, there must be a scalar function $\Phi(\vec{r})$ such that $\vec{J} = -\vec{\nabla}\Phi$. Prove this statement.

b) Show that the potential inside and outside the sphere are given by

$$\Phi_I = -Jr\cos\theta\,, \quad \Phi_O = \frac{Ja^3}{2r^2}\cos\theta.$$

c) The mechanism that keeps the current going must supply the energy dissipated by ohmic work. What is the minimum power required to drive the current?

In this problem, it is assumed that the magnetic fields generated by the currents are negligible, which is valid if the mean free path of the carriers is short enough in the medium. The expression for the gradient in spherical coordinates is

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$