Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented separately in an answer book or on individual sheets of paper. Make sure you clearly indicate who you are and the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.
**Problem 1**

Three particles in a row with masses \( m_1, m_2, \) and \( m_3 \) are connected to each other and to the walls around by identical massless springs, as shown in Fig.1. The spring constant and the equilibrium length are \( k \) and \( a \) respectively, and the distance between walls is \( 4a \). For the one-dimensional motion of all particles:

(a) Construct the Lagrangian of the system.
(b) Derive the Lagrange equations of motion.
(c) Find the eigenfrequencies of harmonic oscillations if the masses of all three particles are equal, \( m_1 = m_2 = m_3 = m \). Describe the character of the motion for each eigenmode.
(d) Find the eigenfrequencies if \( m_1 = m_3 = m \) and \( m_2 = M \).

![Fig.1](https://via.placeholder.com/100)

**Problem 2**

The one-dimensional harmonic oscillator can be studied using a complex variable \( \alpha \) that encompasses both position and momentum at once:

\[
H = \frac{p^2}{2m} + \frac{m \omega^2 x^2}{2}; \quad \alpha = \sqrt{\frac{m\omega}{2}} x + i \sqrt{\frac{1}{2m\omega}} p.
\]

(a) Show that the Poisson brackets for the new variable \( \alpha \) and its complex conjugate \( \alpha^* \) are

\[
[\alpha, \alpha] = [\alpha^*, \alpha^*] = 0 \quad \text{and} \quad [\alpha, \alpha^*] = -i.
\]

(b) Express the Hamiltonian \( H \) as a function of \( \alpha \) and \( \alpha^* \).

(c) Derive the equations of motion for the variables \( \alpha \) and \( \alpha^* \) using Hamilton’s equations of motion, and show that \( \dot{\alpha} = -i \omega \alpha \) and \( \alpha(t) = \alpha(0) \exp(-i \omega t) \).

(d) Obtain the solutions \( x(t) \) and \( p(t) \) of the harmonic oscillator for the given initial values \( x(0) \) and \( p(0) \) using methods outlined in this problem.

**Hint:** The Poisson bracket \( [f, g] \) of two functions \( f \) and \( g \) is defined in classical mechanics as \( [f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \), where \( x \) and \( p \) are the generalized coordinate and momentum. Poisson brackets play the same role in classical mechanics as operator commutators in quantum mechanics.
Problem 3

A particle of mass $m$ is trapped in the field of the “spherical potential well” of radius $R_0$:

$$U(r) = \begin{cases} -U_0, & r \leq R_0, \\ 0, & r > R_0. \end{cases}$$

where $U_0$ is a positive constant. For this central potential, the total energy $E$ and angular momentum $L$ are integrals of motion.

(a) Write the Hamiltonian describing the radial motion of the particle, and find the relationship between $E$ and $L$ values required to keep the trajectory inside the sphere of radius $R_0$.

(b) Describe particle trajectories at different values of $E$ and $L$: calculate the radius $r_c = r_c(E, L)$ of the closest approach to the center of the sphere, and the angle $\Theta = \Theta(E, L)$ of reflection from the surface of the potential well at $R = R_0$.

(c) Establish conditions necessary for circular motion. Are these circular trajectories stable?

(d) Find a relationship between $E$ and $L$ for closed trajectories.

Problem 4

An excited diatomic molecule, moving with the velocity $V$ in the Laboratory Frame (LF), decays into two identical atoms. The decay process is isotropic in the Center of Mass Frame (CMF), where the speeds of the atoms are equal to $v_0$. Calculate the atomic angular distribution function $\varrho(\Theta)$ in the LF, where $\Theta$ is the angle between the LF atomic velocity vectors.

Hint: The function $\varrho(\Theta)$ gives the probability density to detect the angle $\Theta$ between atomic velocity vectors in the LF, and it has to be normalized according to a standard rule: $\int_0^\pi \varrho(\Theta) \frac{1}{2} \sin \Theta d\Theta = 1.$