Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or individual sheets of paper. Make sure you clearly indicate who you are, and the problem you are answering on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.
Problem 1. A particle of mass $m$ moves in the central potential field with the potential energy $U(r) = \alpha r^2$, where $r$ is the distance from the particle to the field center; $\alpha$ is a positive constant.

(a) Construct the Lagrangian of the radial motion of this particle for a given value of the angular momentum $L_0$.

(b) Using the Lagrange equation for the radial motion determine conditions, when the $r$-value does not change in time. Find the value $r_e$ of this equilibrium radius as a function of $L_0, m$ and $\alpha$.

(c) Calculate the frequency of small radial oscillations around the equilibrium radius $r_e$, using leading terms of Taylor’s expansion of the effective potential of the radial motion at $r_e$.

Problem 2. The ends of a light rod, with a mass attached to its center, can slide without friction on a circular wire which is made to rotate about a fixed, vertical diameter with angular velocity $\omega$. Show that, if there is a steady state of motion in which the rod is not horizontal, its inclination $\theta$ to the horizontal is given by

$$\cos(\theta) = \left(\frac{g}{a\omega^2}\right)$$

where $a$ is the distance of the rod’s center from the center of the circular wire.

Problem 3. In a space-time diagram (with axes $x$ and $ct$ using standard notation), the world lines of two free particles A and B are given by $x=0$ and $x=vt$ respectively. Sketch these world lines as well as the world line of a photon leaving A (at the event $E \equiv (0, c\tau)$) and meeting B at the event $F$ in the same space-time diagram, assuming $\tau > 0$. Find the space-time coordinates of $F$. Sketch the world lines in a separate space-time diagram with respect to B (i.e., with axes $x'$ and $ct'$ where $t'$ is B’s proper time). Now find the space-time coordinates of the event $F$ here and show that the ratio of the photon frequency $\nu$ as observed by A, to the frequency $\nu'$ as observed by B is given by

$$\frac{\nu'}{\nu} = \sqrt{\frac{c - v}{c + v}}$$

Hint: You may assume that the number of wavelengths between the points of emission and observation of a photon is a relativistic invariant.
Problem 4. The Lagrangian for a central force is given by

\[ L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r), \]

where \( U(r) \) is the potential energy.

(a) Using Lagrange’s equations for \( r \) and \( \theta \) find the central force equations of motion.

(b) Using the integrals of motion of the equations of motion from part (a) derive the equation for the central force \( F(r) = -\frac{\partial U(r)}{\partial r} \) for a given trajectory \( r = r(\theta(t)) \):

\[ F(r) = -\frac{J^2}{m} u^2 (\frac{d^2 u}{d\theta^2} + u), \]

where \( J = mr^2 \dot{\theta} = \text{constant} \) is the angular momentum and \( u = 1/r \).

(c) Using the orbital equation from part (b) find the force \( F(r) \) which results in the orbit of a particle given by

\[ r = a(1 + \cos \theta), \]

were \( a \) is a positive constant.