**Preliminary Exam: Classical Mechanics, Monday January 8, 2018. 9:00-12:00**

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a separate blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. A particle of mass $m$ is subjected to two forces: a central force $\mathbf{F}_1$ and a frictional force $\mathbf{F}_2$ with $\mathbf{F}_1 = \mathbf{r}f(r)/r$, and $\mathbf{F}_2 = -\lambda\mathbf{v}$ where $\mathbf{v}$ is the velocity of the particle and $\lambda$ is a positive constant.

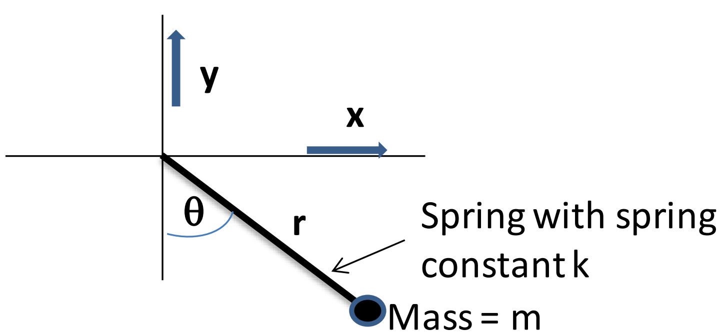
   (**a**) If the particle has initial angular momentum $\mathbf{J}_0$ about $\mathbf{r} = 0$ at time $t = 0$, find an expression that gives the angular momentum as a function of time $t$, expressed if necessary as an integral that you would not then need to evaluate.

   (**b**) With $\lambda = 0$, with $f(r) = -\alpha/r^2$ where $\alpha$ is a positive constant, and with an initial angular momentum $\mathbf{J}_0$ about $\mathbf{r} = 0$ at $t = 0$, find the second-order nonlinear differential equation that determines the function $r(t)$ in terms of $J_0$ and the other constants in the problem. For the special case of a circular orbit with $r = R$ where $R$ is a constant, show that Kepler's Law that $T^2$ is proportional to $R^3$ is valid in this case, where $T$ is the period of the orbit.

2. A spring pendulum is composed of a compact body of mass $m$ suspended from a fixed point by means of a massless spring of spring constant $k$ as shown in the figure. With no weight on the spring its length is $\ell$. Assume that the motion is confined to a vertical plane.

   (**a**) Write the Lagrangian for the system and derive the equation of motion.

   (**b**) Solve the equations of motion (i.e. find $r(t)$ and $\theta(t)$ as a function of time $t$) in the approximation of small angular and small radial displacements from equilibrium. Treat the mass as a point object so that rotation of the mass need not be considered.



3. Consider a uniform, rigid rod of mass $m$ that is placed with one end against a vertical wall and the other (lower) end on a horizontal floor. Both the wall and the floor are frictionless and the rod is free to move in a vertical plane containing it, perpendicular to the wall. The initial angle of inclination of the rod to the horizontal is $\alpha$.

   (**a**) Write down an equation or equations of motion when it is released from rest, while the rod is in contact with both the wall and the floor.

   (**b**) Find the angle (to the horizontal) at which the rod loses contact with the vertical wall.

   (**c**) Write down an equation or equations of motion after the rod has lost contact with the wall and deduce the horizontal velocity component of the center of mass of the rod when it is about to hit the floor.

4. A uniform rigid rod of length $2a$ and mass $m$ hangs in a horizontal position being supported by two vertical strings, each of length $l$ attached to its ends. The other ends of the strings are attached to a ceiling so that the rod and the strings are all in a vertical plane. The rod is given an angular velocity $\omega$ about the vertical axis through its center. Assume that the strings are massless and remain tight during the motion.

   (**a**) Write down the kinetic and potential energy of this system at a twist angle $\theta$ through which the rod has turned.

   (**b**) Find its angular velocity $\dot{\theta}$ when it has turned through any twist angle $\theta$.

   (**c**) Show that its center will rise through a distance $a^2\omega^2/6g$ before coming to instantaneous rest.

   (**d**) Find an upper bound on $\omega$ for such a classical turning point to exist.