

# Quantum and Statistical Mechanics Prelim

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## Statistical Mechanics Questions

1.

a) If  $I=5/2$ , then the particles are fermions and subsequently obey fermi statistics. Thus, there is only one way to arrange four particles in four states when no two particles can occupy the same state.

b) If  $I=2$ , then the particles are bosons and obey bose statistics, and now the particles can be in the same state. We first start with all four particles in one state:

$$C(4, 1) \times 1$$

where  $C(4, 1)$  is the number of combinations of 1 particle in 4 states:  $C(N, n) = \frac{N!}{(N-n)!n!}$ . In other words, it is the number of ways to arrange one particle in one of four states. The 1 comes from the number of ways to arrange the remaining particles in that one level.

We then proceed by dividing the particles among two states:

$$C(4, 2) \times 3$$

Again, the combination is the number of ways to arrange two particles with one in each of two states and three is the number of ways of arranging the remaining particles in those two states.

We repeat this procedure until we reach the possibility of having one particle in each state and we get:

$$\begin{aligned} C(4, 1) \times 1 + C(4, 2) \times 3 + C(4, 3) \times 3 + C(4, 4) \times 1 \\ = 4 + 18 + 12 + 1 = \boxed{35} \end{aligned}$$

Notice that the coefficients of the combinations are nothing other than the elements of Pascal's Triangle. The general formula for calculating the number of ways of arranging  $N$  bosons in  $N$  states is:

$$\sum_{n=1}^N C(N, n)C(N-1, n-1)$$

For small  $N$ , it's easy to enumerate the number of states by drawing all possible configurations. However, if  $N$  is large, it is much easier to use the formula above.

2.

We have the potential of the particles in the ring:

$$u_{ij}(r) = \frac{-\epsilon}{r^6}$$

for  $\sigma \leq r \leq L/2$

This basically means that the particles are hard spheres with diameters equal to  $\sigma$ . This is why the particles cannot get closer than  $\sigma$  to each other; the potential is infinite.

a) To calculate the total potential energy  $U$  of this system, we proceed as follows:

$$\begin{aligned} U &= \sum_{i=1}^N \sum_{j=0}^{i-1} u_{ij}(r) \rightarrow - \left( \frac{N}{L} \right)^2 \epsilon \left[ \int_{\sigma}^{L/2} \int_0^{r_2-\sigma} \frac{dr_1 dr_2}{(r_1 - r_2)^6} + \int_{-\sigma}^{-L/2} \int_0^{r_2-\sigma} \frac{dr_1 dr_2}{(r_1 - r_2)^6} \right] \\ &= \left( \frac{N}{L} \right)^2 \frac{\epsilon}{5} \left[ \int_{\sigma}^{L/2} \frac{dr_2}{(r_1 - r_2)^5} \Big|_0^{r_2-\sigma} + \int_{-\sigma}^{-L/2} \frac{dr_2}{(r_1 - r_2)^5} \Big|_0^{r_2-\sigma} \right] \\ &= \frac{N^2 \epsilon}{5L^2} \left[ \int_{\sigma}^{L/2} \left( -\frac{1}{\sigma^5} + \frac{1}{r_2^5} \right) dr_2 + \int_{-\sigma}^{-L/2} \left( -\frac{1}{\sigma^5} + \frac{1}{r_2^5} \right) dr_2 \right] \\ &= \boxed{\frac{N^2 \epsilon}{5L^2} \left( \frac{1}{2\sigma^4} - \frac{8}{L^4} \right)} \end{aligned}$$

### 3.

a) Let  $N_B$  be the number of bosons. At the critical temperature, the boson gas undergoes a phase transition due to ionization and the two constituent fermions become free. Thus, let  $N_f = 2N_B$  be the number of fermions after ionization. We will treat this system as a lattice gas by requiring that each molecule occupy a lattice site in the "bosonic space" and that each fermion occupy a lattice site in "fermionic space". Thus,  $n_B = \frac{V}{v_B}$  is the number of lattice sites in "bosonic space" and  $n_f = \frac{V}{v_f}$  is the number of lattice sites in "fermionic space".

The multiplicity is the number of ways in which the bosons can be arranged in  $n_B$  lattice sites and is given by:

$$g(N_B, n_B) = \frac{n_B!}{N_B!(n_B - N_B)!}$$

The entropy is given by

$$\begin{aligned} S_B &= k \ln(g_B) = k [n_B \ln(n_B) - n_B - (n_B - N_B) \ln(n_B - N_B) + (n_B - N_B) - N_B \ln(N_B) + N_B] \\ &= k [N_B \ln(n_B) - N_B \ln(N_B)] = k \ln \left( \frac{n_B}{N_B} \right)^{N_B} = \boxed{k \ln \left( \frac{V}{v_B N_B} \right)^{N_B}} \end{aligned}$$

Note that we have made use of Stirling's approximation and the fact that  $n_B \gg N_B$ . The entropy after ionization is

$$\boxed{S_F = k \ln \left( \frac{V}{2v_f N_B} \right)^{2N_B}}$$

b) The latent heat  $\Delta q$  is given by:

$$\Delta q = T(s_f - s_B)$$

where  $s_f$  and  $s_B$  are the entropies per particle of the fermions and bosons respectively.

$$\begin{aligned} s_f - s_B &= \frac{kT_C}{N_B} \left[ \ln \left( \frac{V}{2v_f N_B} \right)^{2N_B} - \ln \left( \frac{V}{v_B N_B} \right)^{N_B} \right] \\ \rho &= \frac{2N_B}{V} \rightarrow N_B = \frac{\rho V}{2} \end{aligned}$$

$$\begin{aligned}\Delta q &= \frac{2kT_C}{\rho V} \left[ \ln \left( \frac{1}{\rho v_f} \right)^{\rho V} - \ln \left( \frac{2}{\rho v_B} \right)^{\rho V/2} \right] \\ &= 2kT_C \ln \left( \frac{(\rho v_B)^{1/2}}{\sqrt{2}\rho v_f} \right) = \boxed{kT_C \ln \left( \frac{v_B}{2\rho v_f^2} \right)}\end{aligned}$$

c) The binding energy  $B$  of the bosons is simply the energy needed to ionize the molecule. But this is exactly the latent heat because the latent heat is the amount of energy required to induce phase transition without changing the temperature of the gas. Thus,

$$B = -\Delta q$$

## Quantum Mechanics Questions

### 1.

We can write the outgoing  $|+1/2\rangle_z$  and  $|-1/2\rangle_z$  states from A as linear combinations of  $|\pm 1/2\rangle_x$  or  $|\pm 1/2\rangle_y$ . Since the Stern-Gerlach apparatus at point D is oriented in the y direction, we shall decompose the  $|-1/2\rangle_z$  state at A into its y component:

$$|-1/2\rangle_z = \frac{i}{\sqrt{2}} (|-1/2\rangle_y - |+1/2\rangle_y)$$

This is basically saying that because the particle from A is polarized in the negative z direction, we have complete ignorance of its y polarization.

Thus, at C, we have the *amplitude* contribution from B:  $|-1/2\rangle_y$  and the contribution from A:  $1/2|-1/2\rangle_y$  and  $1/2|+1/2\rangle_y$ . Thus, after normalizing the resulting state we have:

$$3/4|-1/2\rangle_y \text{ and } 1/4|+1/2\rangle_y$$

So we would expect to find two traces on the screen, one trace corresponding to  $|+1/2\rangle_y$  with relative intensity 1/4, and the other trace corresponding to  $|-1/2\rangle_y$  with relative intensity 3/4.

Incidentally, if the second filter was tilted with respect to the first by some angle  $\alpha$ , we would have to decompose  $|+1/2\rangle_z$  into its  $\alpha$  components:

$$|+1/2\rangle_z = \cos(\alpha/2)|+1/2\rangle_\alpha + \sin(\alpha/2)|-1/2\rangle_\alpha$$

So the probability of finding particles in the state  $|-1/2\rangle_\alpha$  is  $\alpha < -1/2|+1/2\rangle_z$  and the probability of finding particles in the state  $|+1/2\rangle_\alpha$  is  $\alpha < +1/2|+1/2\rangle_z$ .

## 2.

The unperturbed Hamiltonian is given by  $H_0 = \sum_{i=1}^N E_i |i\rangle\langle i|$  and the system is initially found in the  $|1\rangle$  state. The interaction has the form

$$W(t) = \sum_{i,j=1,i \neq j}^N \frac{\gamma}{1 + (B - B_{ij})^2/B_{12}^2} e^{i\omega t} |i\rangle\langle j| \\ + \sum_{i,j=1,i \neq j}^N \frac{\gamma}{1 + (B - B_{ji})^2/B_{21}^2} e^{-i\omega t} |j\rangle\langle i|$$

where  $\gamma$  and  $B_{ij}$  are real. Notice that the second term is the hermitian conjugate of the first. The parts to this problem are ambiguous in my opinion, and I will therefore go through the solution in my own order. The probability coefficient of the  $i^{\text{th}}$  state is given by

$$c_i(t) = e^{iE_i t/\hbar} \langle i|\psi\rangle$$

We can expand any wavefunction in terms of the basis states as follows

$$|\psi(t)\rangle = \sum_k c_k(t) |k\rangle = \sum_k e^{iE_k t/\hbar} \langle k|\psi\rangle |k\rangle$$

Now, we shall use time independent perturbation theory to see how this coefficient changes when we introduce the interaction. Also, I assume that  $B_{ij} = B_{ji}$ , i.e. the matrix is symmetric.

$$\langle k|\psi\rangle = \frac{1}{i\hbar} \int_0^t \langle k|W|1\rangle e^{\frac{it'(E_k - E_1)}{\hbar}} dt' \\ \langle k|W|1\rangle = \sum_{i=2}^N \frac{\gamma}{1 + (B - B_{ij})^2/B_{12}^2} \int_0^t \left( e^{-i\omega t'} + e^{i\omega t'} \right) e^{\frac{it'(E_k - E_1)}{\hbar}} dt' \\ = A \int_0^t \left( e^{it'(\omega_k - \omega_1 - \omega)} + e^{it'(\omega_k - \omega_1 + \omega)} \right) dt' \\ = A \left[ \frac{1}{i(\omega_k - \omega_1 - \omega)} \left( e^{it(\omega_k - \omega_1 - \omega)} - 1 \right) + \frac{1}{i(\omega_k - \omega_1 + \omega)} \left( e^{it(\omega_k - \omega_1 + \omega)} - 1 \right) \right]$$

where

$$A = \frac{\gamma}{1 + (B - B_{ij})^2/B_{12}^2}$$

Since we want to couple the  $|1\rangle$  and  $|2\rangle$  states, we let  $k = 2$  above. However, notice that  $\omega_2 > \omega_1$  and  $\omega \geq 0$ . This implies that the second

term in the expression above is not physical at resonance. Another way of saying this is that the system undergoes absorption rather than stimulated emission because it absorbs a photon. Letting  $B = B_{12}m$  we obtain

$$\begin{aligned} \langle 2|\psi \rangle &= \frac{-i\gamma}{\hbar(\omega_2 - \omega_1 - \omega)} \left( e^{it(\omega_2 - \omega_1 - \omega)} - 1 \right) \\ |\langle 2|\psi \rangle|^2 &= \frac{\gamma^2}{\hbar^2 (\omega_2 - \omega_1 - \omega)^2} \left( 2 - e^{it(\omega_2 - \omega_1 - \omega)} - e^{-it(\omega_2 - \omega_1 - \omega)} \right) \\ &= \frac{\gamma^2}{\hbar^2 (\omega_2 - \omega_1 - \omega)^2} (2 - 2\cos((\omega_{21} - \omega)t)) \end{aligned}$$

where  $\omega_{21} = \omega_2 - \omega_1$ . Now, using the following trigonometric identity

$$\sin^2(\theta) = 1/2 - 1/2\cos(2\theta)$$

we obtain

$$|\langle 2|\psi \rangle|^2 = \frac{4\gamma^2}{\hbar^2(\omega_{21} - \omega)^2} \sin^2 \left( \frac{(\omega_{21} - \omega)t}{\hbar} \right)$$

If  $\gamma = \frac{(\omega_{21} - \omega)\hbar}{2}$  then we recover the Rabi formula

$$|\langle 2|\psi \rangle|^2 = \sin^2 \left( \frac{\gamma t}{\hbar} \right)$$

### 3.

A hydrogen atom is oriented along the z-axis in a constant external electric field.

a) The perturbation  $W$  due to the field is

$$W = -eE\vec{Z} = -eE\cos(\theta)$$

b) The first-order correction to the energy of the 1s level in perturbation theory is given by

$$E^{(1)} = \langle 1|W|1 \rangle = 0$$

The reason this is zero is because the angular part of the hydrogen groundstate wavefunction is even, while the perturbation,  $\cos(\theta)$ , is odd. Thus, the resulting integral is odd over the interval from 0 to  $2\pi$ . This result makes sense because the electron cloud in the groundstate of the hydrogen atom is symmetric. Consequently, there is no permanent dipole that interacts with the electric field and no energy shift. The second-order energy shift represents the interaction of the electric field with the *induced* dipole moment.

c) The energy shift is given by  $\Delta E = -\vec{E} \cdot \vec{D}$  where  $\vec{D}$  is the dipole moment. The polarizability is given by  $\alpha = \frac{|\vec{D}|}{|\vec{E}|}$ . We shall start by writing the correction to the energy in second order:

$$E_{|100\rangle}^{(2)} = \sum_{m \neq 1} = \frac{|\langle m10|W|100 \rangle|^2}{E_1^{(0)} - E_m^{(0)}}$$

but if we assume that  $E_1 - E_n \approx E_1 - E_2$ , this reduces to

$$= \frac{1}{E_1^{(0)} - E_2^{(0)}} \sum_{m \neq 1} \langle 100|W|m10 \rangle \langle m10|W|100 \rangle$$

Since these basis vectors are complete, we can use the closure relation to reduce the expression above. Recall that completeness implies that

$$\sum_m |m10 \rangle \langle m10| = 1$$

Thus,

$$\begin{aligned}
E_{|100\rangle}^{(2)} &= \frac{1}{E_1^{(0)} - E_2^{(0)}} \langle 100|W^2|100\rangle \\
&= \frac{e^2 E^2}{E_1 - E_2} \langle 100|r^2 \cos^2(\theta)|100\rangle \\
&= \frac{e^2 E^2}{E_1^{(0)} - E_2^{(0)}} \int_0^\infty R_{10}^*(r) r^2 R_{10}(r) dr \int_0^\pi \int_0^{2\pi} Y_0^{0*}(\cos(\theta)) \cos(\theta) Y_0^0(\cos(\theta)) \sin(\theta) d\theta d\phi \\
&= \frac{3a_0^2 e^2 E^2}{E_1 - E_2} \frac{1}{2} \int_0^\pi \cos^2(\theta) \sin(\theta) d\theta = \frac{-3a_0^2 e^2 E^2}{2(E_1 - E_2)} \frac{1}{3} (-1 - 1) \\
&= \frac{a_0^2 e^2 E^2}{E_1 - E_2} \\
\alpha &= \frac{-a_0^2 e^2}{E_1 - E_2} \quad ; \quad E_2 = E_1/4 \\
\alpha &= \frac{-4a_0^2 e^2}{3E_1^{(0)}} \quad ; \quad E_1^{(0)} = \frac{e^2}{2n^2 a_0}
\end{aligned}$$

$$\boxed{\alpha = \frac{-8a_0^3}{3}}$$

4.

a) The asymptotic form the partial wave solution can be obtained by using the asymptotic form of the Bessel functions given on the first page.

$$j_l(r) \rightarrow \frac{\sin(r - \frac{l\pi}{2})}{r} \quad ; \quad n_l(r) \rightarrow -\frac{\cos(r - \frac{l\pi}{2})}{r}$$

$$u(r, \theta)_p \rightarrow \sum_{l=0} A_l \left[ \frac{\cos(\delta_l) \sin(kr - \frac{l\pi}{2})}{kr} + \frac{\sin(\delta_l) \cos(kr - \frac{l\pi}{2})}{kr} \right] P_l(\cos(\theta))$$

Assuming that  $A_l = (2l + 1) i^l e^{i\delta_l}$ , and using angle addition relations, we get

$$u(r, \theta)_p = \sum_{l=0} \frac{(2l + 1) i^l e^{i\delta_l}}{kr} \sin(kr - \frac{l\pi}{2} + \delta_l) P_l(\cos(\theta))$$

Now we have to match this to the scattered solution:

$$u(r, \theta)_s \rightarrow C \left[ e^{ikz} + \frac{1}{r} f(\theta) e^{ikr} \right]$$

We make the following substitutions:

$$f(\theta) = \frac{1}{2ik} \sum_{l=0} (2l + 1) (e^{2i\delta_l} - 1) P_l(\cos(\theta))$$

$$e^{ikz} = \sum_{l=0} (2l + 1) i^l j_l(kr) P_l(\cos(\theta))$$

This gives us:

$$\begin{aligned} u(r, \theta)_s &\rightarrow C \left( \sum_{l=0} \frac{(2l + 1) i^l}{kr} \sin(kr - \frac{l\pi}{2}) P_l(\cos(\theta)) + \sum_{l=0} \frac{(2l + 1)}{2ikr} e^{ikr} (e^{2i\delta_l} - 1) P_l(\cos(\theta)) \right) \\ &= \sum_{l=0} \frac{(2l + 1) i^l}{kr} e^{i\delta_l} \left( e^{-i\delta_l} \sin(kr - \frac{l\pi}{2}) + e^{i(kr - \frac{l\pi}{2})} \sin(\delta_l) \right) P_l(\cos(\theta)) \end{aligned}$$

Consider the following identities:

$$e^{i\delta_l} \sin(kr - \frac{l\pi}{2}) = \cos(\delta_l) \sin(kr - \frac{l\pi}{2}) - i \sin(\delta_l) \sin(kr - \frac{l\pi}{2})$$

$$e^{i(kr - \frac{l\pi}{2})} \sin(\delta_l) = \cos(kr - \frac{l\pi}{2}) \sin(\delta_l) + i \sin(kr - \frac{l\pi}{2}) \sin(\delta_l)$$

Adding these two equations yields  $\sin(\delta_l + kr - \frac{l\pi}{2})$ . Using this in the expression for  $u(r, \theta)_s$  gives us

$$u(r, \theta)_s \rightarrow \sum_{l=0} \left[ \frac{(2l+1)i^l}{kr} e^{i\delta_l} \sin(kr - \frac{l\pi}{2} + \delta_l) P_l(\cos(\theta)) \right]$$

Thus, we see that the two solutions are compatible, but only if

$$A_l = (2l+1)i^l e^{i\delta_l} \quad ; \quad f(\theta) = \frac{1}{2ik} \sum_{l=0} (2l+1)(e^{2i\delta_l} - 1) P_l(\cos(\theta))$$

b) We start by writing the time independent radial Schrodinger Equation for  $r < a$ :

$$\begin{aligned} \frac{-\hbar^2}{2m} \nabla_r^2 R(r) - V_0 R(r) &= ER(r) \\ \nabla_r^2 R(r) &= \frac{-2m}{\hbar^2} (E + V_0) R(r) \end{aligned}$$

Since  $E = \frac{\hbar^2 k^2}{2m}$ , we have

$$\nabla_r^2 R = - \left( k^2 + \frac{2mV_0}{\hbar^2} \right) R(r)$$

$\alpha^2 = k^2 + \frac{2mV_0}{\hbar^2} =$  the wavenumber of the particle in the spherical well

For the region  $r < a$ , the solution is given by

$$R(r) = A_l (\cos(\delta_l) j_l(\alpha r) - \sin(\delta_l) n_l(\alpha r))$$

We require that this solution does not blow up at the origin, and since  $R(0) = \infty$  and  $n_l(0) = \infty$ ,

$$\delta_l = m\pi \quad ; \quad m = 0, 1, 2, \dots$$

$$R(r) = A_l (-1)^{m+1} j_0(\alpha r)$$

Taking the logarithmic derivative to simultaneously satisfy boundary conditions on the wavefunction and its derivative yields:

$$\frac{R'}{R} = \frac{\alpha j_0'(\alpha a)}{j_0(\alpha a)}$$

Thus,

$$\gamma_l = \frac{R'}{R} \Big|_{r=a} = \frac{\alpha j_l'(\alpha a)}{j_l(\alpha a)}$$

$$\alpha = \sqrt{k^2 + \frac{2mV_0}{\hbar^2}}$$

5.

The Hamiltonian of our system is

$$H = \frac{1}{2I} \mathbf{L}^2 + \mu L_3$$

where  $I$  is the moment of inertia.

a) The spectrum of the Hamiltonian can be obtained from the following eigen equation:

$$H|\psi\rangle = E|\psi\rangle = E|lm\rangle$$

where  $l$  and  $m$  represent the orbital and magnetic quantum numbers respectively.

$$H|lm\rangle = \left( \frac{L^2}{2I} + \mu L_z \right) |lm\rangle = \boxed{\left( \frac{\hbar^2}{2I} l(l+1) + \mu m \hbar \right) |lm\rangle}$$

b) The commutation relations are calculated as follows:

$$[L_z, L_\pm] = [L_z, L_x \pm iL_y] = [L_z, L_x] \pm i[L_z, L_y]$$

$$i\hbar L_y \pm \hbar L_x = \pm \hbar(L_x \pm iL_y) = \boxed{\pm \hbar L_\pm}$$

We show that  $L_-$  acts as a lowering operator by making use of the commutation above in the following operation:

$$L_z(L_-|lm\rangle) = L_-L_z|lm\rangle - \hbar L_-|lm\rangle$$

$$L_-m\hbar|lm\rangle - \hbar L_-|lm\rangle = \boxed{\hbar(m-1)(L_-|lm\rangle)}$$

Thus, we see that  $L_-|lm\rangle$  is an eigenstate of  $L_z$  with eigenvalue  $\hbar(m-1)$ . Since the eigenvalue of  $L_z$  is  $m\hbar$ , it must be that  $L_-$  lowers the eigenvalue of  $L_z$  by one. Similarly, for  $L_+$ , we have

$$\begin{aligned} L_z(L_+|lm\rangle) &= L_+L_z|lm\rangle + \hbar L_+|lm\rangle = m\hbar L_+|lm\rangle + \hbar L_+|lm\rangle \\ &= \boxed{\hbar(m+1)(L_+|lm\rangle)} \end{aligned}$$

Thus,  $L_+$  raises the eigenvalue of  $L_z$  by one.

c) Now we consider the following operation:

$$\begin{aligned}
H \left( \frac{L_{\pm}}{\hbar\sqrt{2l}} \right) |lm\rangle &= \frac{1}{\hbar\sqrt{2l}} \left( \frac{L^2}{2I} + \mu L_z \right) L_{\pm} |l, -l\rangle \\
&= \frac{1}{\hbar\sqrt{2l}} \left( L_{\pm} \left( \frac{1}{2I} L^2 + \mu L_z \right) |l, -l\rangle + [H, L_{\pm}] |l, -l\rangle \right) \\
&\quad [H, L_{\pm}] = \mu [L_z, L_{\pm}] = \pm \hbar \mu L_{\pm} \\
H \left( \frac{L_{\pm}}{\hbar\sqrt{2l}} \right) |lm\rangle &= \frac{1}{\hbar\sqrt{2l}} \left( L_{\pm} \left( \frac{\hbar^2}{2I} l(l+1) - \mu \hbar l \right) |l, -l\rangle \pm \mu \hbar L_{\pm} |l, -l\rangle \right) \\
&= \boxed{\frac{1}{\hbar\sqrt{2l}} L_{\pm} |l, -l\rangle \left( \frac{\hbar^2}{2I} l(l+1) - \mu \hbar (l \pm 1) \right)}
\end{aligned}$$