

Classical Mechanics and Electrodynamics Prelim

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Classical Mechanics Questions

1.)

In a rotating frame, Newton's equation reads:

$$\vec{F}_{eff} = m\vec{a}_r = m\vec{a}_f + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}$$

a) To find the lateral deflection of a projectile that is fired due east from a point on the surface of the Earth at a northern latitude λ , with a speed v_0 , at an angle with respect to the horizontal of α , we must consider each term in the equation above.

The centrifugal term is of order ω^2 and $\omega = \frac{2\pi}{24(3600)} = 7.3 \times 10^{-5} rad/s$, so we'll neglect this term. Since the Earth's rotation is roughly constant, we need not worry about angular acceleration terms. Thus, we now have the following modified equation which includes the external gravitational force and the Coriolis force:

$$\vec{F} = -2m\vec{\omega} \times \vec{v} - m\vec{g}$$

We configure our observer's axes such that the z-axis points along the observer's zenith, the x-axis points due East, and the y-axis points due North. Thus, we have:

$$\vec{\omega} \times \vec{v} = (-\omega \cos \lambda \hat{x} + \omega \sin \lambda \hat{z}) + (\dot{x} \hat{x} + \dot{y} \hat{y} + \dot{z} \hat{z}) = -\omega \dot{y} \cos \lambda \hat{z} + \omega \dot{z} \cos \lambda \hat{y} + \omega \dot{x} \sin \lambda \hat{y} - \omega \dot{y} \sin \lambda \hat{x}$$

This gives us the following coupled equations:

$$\ddot{x} = 2\dot{y}\omega \sin \lambda$$

$$\ddot{y} = -2\omega \cos \lambda \dot{z} - 2\omega \sin \lambda \dot{x}$$

$$\ddot{z} = -g + 2\omega \dot{y} \cos \lambda$$

This is a difficult problem because the Coriolis Effect will tend to change the x position of the projectile which will then change the y and z positions, which in turn deflects the projectile again in x . This cyclic process is difficult to deal with so we'll make it more tractable by assuming that the z and y motion are *not* affected by the Coriolis force. Therefore, we can neglect the second term in the last equation above.

$$z = v_0 t + \frac{1}{2} g t^2$$

We can determine how long the projectile is in flight by setting the final height to be zero:

$$0 = T(v_0 \sin \alpha - \frac{gT}{2}) \rightarrow T = \frac{2v_0 \sin \alpha}{g}$$

Since $\dot{y} = v_0 \cos \alpha t$, we can now determine an expression for the velocity in the x direction:

$$\dot{x} = 2v_0 \cos \alpha \omega \sin \lambda t$$

Integrating, and plugging in the expression for the time of flight yields

$$\Delta x = \frac{4\omega v_0^3 \sin^2 \alpha \cos \alpha \sin \lambda}{g^2}$$

b) If $v_0 = 300$ m/s, $\alpha = 30$ deg, and $\lambda = 60$ deg, the lateral deflection is $\Delta x \approx 15$ m.

2.)

There is a central force given by $F(r) = -\frac{k}{r^2} - \frac{k'}{r^4}$.

a) The minimum value of the angular momentum, such that circular orbits exist can be found by setting the effective force equal to zero. This is true because a circular orbit is defined by a constant radius, so we must find a local maximum or minimum in the effective potential where the slope of the tangent line is zero.

$$V_{eff} = V + \frac{l^2}{2mr^2}$$

where V is the potential given by $V = -\int F dr$, and the second term represents the centrifugal barrier for nonzero angular momenta. We have:

$$V = -\int \left(-\frac{k}{r^2} - \frac{k'}{r^4} \right) dr = -\left(\frac{k}{r} + \frac{k'}{3r^3} \right)$$

$$V_{eff} = -\frac{k}{r} - \frac{k'}{3r^3} + \frac{l^2}{2mr^2}$$

$$\frac{\partial V_{eff}}{\partial r} = -\frac{k}{r^2} - \frac{k'}{r^4} + \frac{l^2 r}{m} = 0$$

$$k' + kr^2 - \frac{l^2 r}{m} = 0$$

$$kr^2 - \frac{l^2 r}{m} + k' = 0$$

$$r = \frac{l^2}{2km} \pm \frac{1}{2km} \sqrt{l^4 - 4kk'm^2} = \frac{1}{2km} \left(l^2 \pm \sqrt{l^4 - 4kk'm^2} \right)$$

In order for the radius to remain real, we see that $l^4 \geq 4kk'm^2$, so that the minimum angular momentum needed for circular orbit is:

$$l_{min} = (4kk'm^2)^{\frac{1}{4}}$$

Plugging this in for the expression for the radius yields:

$$r_{circ} = \frac{l_{min}^2}{2km} = \sqrt{\frac{k'}{k}}$$

b) If L is greater than the minimum value, we must consider both solutions:

$$r_+^2 = \frac{l^4}{4k^2m^2} + \frac{1}{4k^2m^2}(l^4 - 4kk'm^2)$$

$$r_-^2 = \frac{l^4}{4k^2m^2} - \frac{1}{4k^2m^2}(l^4 - 4kk'm^2)$$

The condition for a stable orbit is that the effective potential is concave up, because small displacements in r would bring you to a region of larger potential. This is mathematically described by the second derivative of the effective potential being positive. In other words:

$$\frac{2k}{r^3} + \frac{4k'}{r^5} - \frac{3l^2}{mr^4} > 0$$

$$2kr^2 + 4k' - 3l^2r > 0$$

Upon plugging in r_+ and r_- into the equation above, we find that both solutions satisfy the inequality. Thus, there are two solutions for circular orbit when $l > l_{min}$.

3.)

The following solution was provided by graduate students, Leland Aldridge and James Wells.

A cue stick hits a billiard ball of radius R and mass M at a height h above the center of mass. The moment of inertia of a sphere about an axis through its center is $I = \frac{2}{5}MR^2$.

a) To calculate the final velocity of the ball, we first consider the torque on the billiard ball due to the cue stick:

$$\tau = \vec{r} \times \vec{F} = I\vec{\alpha} = \frac{2}{5}MR^2\vec{\alpha}$$

Integrating with respect to time, and keeping in mind that the force is the time rate of change of the momentum, gives us the change in angular velocity of the ball:

$$\Delta\vec{\omega} = \frac{5\vec{r} \times \vec{p}}{2MR^2}$$

Since the ball is initially at rest, the angular velocity immediately after the impulse is given by

$$\omega_0 = \frac{5RMV_0 \sin \theta}{2MR^2} = \frac{5V_0 \sin \theta}{2R} = \frac{5h}{2R^2}V_0$$

Once the initial impulse is complete, the frictional force is the only force acting on the billiard ball and will tend to slow the translational velocity of the ball, while providing a torque which causes the ball to increase its angular velocity. We may apply our kinematic equations as follows:

$$V(t) = V_0 - \frac{f}{M}t$$

$$\omega(t) = \omega_0 + \frac{\tau}{I}t = \frac{5h}{2R^2}V_0 + \frac{5f}{2MR}t$$

Notice that the lever arm in this case is R , and not h , as friction will exert a torque at the bottom of the ball, a distance R from the center of mass. At the time t_c , the ball will begin to roll without slipping and $V = R\omega$.

$$V(t_c) = V_0 - \frac{f}{M}t_c = R\omega(t_c) = \frac{5h}{2R}V_0 + \frac{5f}{2M}t_c$$

$$t_c = \frac{2M}{7f} \left(1 - \frac{5h}{2R}\right) V_0$$

Substituting the expression for t_c in that for $V(t)$ yields

$$V(t_c) = V_0 - \frac{2}{7} \left(1 - \frac{5h}{2R}\right) V_0 = \left(1 - \frac{2}{7} \left(1 - \frac{5h}{2R}\right)\right) V_0 = \boxed{\frac{5}{7} \left(1 + \frac{h}{R}\right) V_0}$$

Also note that energy conservation is not easily applicable here because the frictional force dissipates some of the ball's energy in its slipping phase.

b) When $V = V_0$, we have

$$1 = \frac{5}{7} \left(1 + \frac{h_c}{R}\right) = 1$$

$$h_c = \left(\frac{7}{5} - 1\right) R = \boxed{\frac{2}{5} R}$$

This is the height at which we must hit the ball so that it immediately begins to roll without slipping and $V = V_0$. Hitting the ball lower than this will cause it to slide under the influence of friction before it starts to roll without slipping. If we hit the ball at a point higher than this, the ball will be spinning too fast and must slow its rotation before rolling without slipping.

c) We see that the final velocity of the ball increases with h so that any height between h_c and R will cause $V > V_0$. Of course, heights above R are not relevant because in this case, the cue stick misses the ball entirely.

4.)

Two identical thin rods of mass M and length l are suspended from a pivot and connected so that they can swing freely in a vertical plane.

a) The kinetic energy is the sum of the center of mass kinetic energy and the rotational energy:

$$T = t_{cm} + T_{rot}$$

where $T_{rot} = \frac{1}{2}I\omega^2$ with $I = \frac{1}{2}ml^2$. The generalized coordinates are θ , the angle from the vertical of the top pendulum, and ϕ , the angle from the vertical of the bottom pendulum. Next, we define the x and y positions of each pendulum in terms of the generalized coordinates:

$$\begin{aligned}x_1 &= \frac{l}{2}\sin\theta & y_1 &= -\frac{l}{2}\cos\theta \\x_2 &= l\sin\theta + \frac{l}{2}\sin\phi & y_2 &= -l\cos\theta - \frac{l}{2}\cos\phi \\x_1 &= \frac{l}{2}\dot{\theta}\cos\theta & y_1 &= \frac{l}{2}\dot{\theta}\sin\theta \\x_2 &= l\dot{\theta}\cos\theta + \frac{l}{2}\dot{\phi}\cos\phi & y_2 &= l\dot{\theta}\sin\theta + \frac{l}{2}\dot{\phi}\sin\phi\end{aligned}$$

The kinetic energy is then

$$\begin{aligned}T &= \frac{m}{2} \frac{l^2}{4} \dot{\theta}^2 \cos^2\theta + \frac{l^2}{4} \dot{\theta}^2 \sin^2\theta + l^2 \dot{\theta}^2 \cos^2\theta + \frac{l^2}{4} \dot{\phi}^2 \cos^2\phi + l^2 \dot{\phi} \dot{\theta} \cos\theta \\&\quad + \cos\phi + l^2 \dot{\theta}^2 \sin^2\theta + \frac{l^2}{4} \dot{\phi}^2 \sin^2\phi + l^2 \dot{\theta} \dot{\phi} \sin\theta \sin\phi + \frac{1}{24} ml^2 (\dot{\theta}^2 + \dot{\phi}^2) \\T &= \frac{m}{2} \left[\frac{l^2}{4} \dot{\theta}^2 + l^2 \dot{\theta}^2 + \frac{l^2}{4} \dot{\phi}^2 + l^2 \dot{\theta} \dot{\phi} (\sin\theta \sin\phi + \cos\theta \cos\phi) + \frac{l^2}{12} (\dot{\theta}^2 + \dot{\phi}^2) \right] \\&= \frac{m}{2} \left[\frac{l^2 \dot{\theta}^2}{3} + l^2 \dot{\theta}^2 + \frac{l^2 \dot{\phi}^2}{3} + l^2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right] \\L &= \frac{ml^2}{2} \left(\frac{4\dot{\theta}^2}{3} + \frac{\dot{\phi}^2}{3} + \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right) - \frac{m}{2} gl (\cos\theta + 2\cos\theta + \cos\phi)\end{aligned}$$

$$\boxed{L = \frac{ml^2}{2} \left(\frac{4\dot{\theta}^2}{3} + \frac{\dot{\phi}^2}{3} + \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right) - \frac{m}{2} gl (3\cos\theta + \cos\phi)}$$

b) We start by calculating the following:

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{4ml^2}{3}\dot{\theta} + \frac{ml^2}{2}\dot{\phi}\cos(\theta - \phi)$$

$$\frac{\partial L}{\partial \theta} = -\frac{ml^2}{2}\dot{\theta}\dot{\phi}\sin(\theta - \phi) + \frac{3mgl}{2}\sin\theta$$

Next, we evaluate Lagrange's equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{aligned} \frac{4ml^2}{3}\ddot{\theta} + \frac{ml^2}{2}\ddot{\phi}\cos(\theta - \phi) - \frac{ml^2\dot{\phi}}{2}\dot{\theta}\sin(\theta - \phi) \\ + \frac{ml^2\dot{\phi}^2}{2}\sin(\theta - \phi) + \frac{ml^2\dot{\theta}\dot{\phi}}{2}\sin(\theta - \phi) - \frac{3mgl}{2}\sin\theta = 0 \end{aligned}$$

For small angles, anything of second order or higher in θ , ϕ , $\dot{\theta}$, or $\dot{\phi}$ is small. Thus, we have

$$\frac{4ml^2}{3}\ddot{\theta} + \frac{ml^2}{2}\ddot{\phi} - \frac{3mgl}{2}\theta = 0$$

Now we'll write Lagrange's equation in the ϕ coordinate:

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{ml^2}{3}\dot{\phi} + \frac{ml^2}{2}\dot{\theta}\cos(\theta - \phi)$$

$$\frac{\partial L}{\partial \phi} = \frac{ml^2}{2}\dot{\theta}\dot{\phi}\sin(\theta - \phi) + \frac{mgl}{2}\sin\phi$$

Using the information above, we find that the coupled equations of motion in θ and ϕ are:

$$\boxed{\frac{ml^2}{3}\ddot{\phi} + \frac{ml^2}{2}\ddot{\theta} - \frac{mgl}{2}\phi = 0}$$

$$\boxed{\frac{4ml^2}{3}\ddot{\theta} + \frac{ml^2}{2}\ddot{\phi} - \frac{3mgl}{2}\theta = 0}$$

Electromagnetism Questions

5.)

The magnitude of the magnetic field from the bar magnet is $B = \frac{k}{4\pi r^2}$ where $k = 4 \times 10^{-5} W$. As the magnet moves toward the coil, the changing magnetic flux through the conducting coil will induce an *emf* given by $emf = -\frac{d}{dt} \Phi_B$.

We start by calculating the magnetic flux. First, note that the flux through the semi-sphere bounded by the ring is the same as the flux through the ring. Thus,

$$\Phi_B = \oint \vec{B} \cdot d\vec{a} = \frac{k}{4\pi} \int \sin \theta d\theta d\phi = \frac{k}{2} \int_0^{\theta_{max}} \sin \theta d\theta$$

The coordinate r is related to x , the horizontal distance from the magnet to the coil and R , the radius of the coil by $r^2 = x^2 + R^2$. Thus,

$$\begin{aligned} \Phi_B &= \frac{k}{2} \cos \theta \Big|_{\theta_{max}}^0 = \frac{k}{2} \left(1 - \frac{x}{\sqrt{x^2 + R^2}} \right) \\ \frac{d\phi}{dt} &= \frac{d\phi}{dx} v = \frac{k}{2} v \left(-\frac{1}{\sqrt{x^2 + R^2}} + \frac{1}{2} \frac{x \cdot 2x}{(x^2 + R^2)^{3/2}} \right) \\ &= \frac{k}{2} \left(\frac{x^2}{(x^2 + R^2)^{3/2}} - \frac{1}{(x^2 + R^2)^{1/2}} \right) \end{aligned}$$

$x = 0$ when the magnet just passes through the center of the coil.

$$\left. \frac{d\phi}{dt} \right|_{x=0} = -\frac{kv}{2R}$$

$$emf = \frac{kv}{2R} = \frac{(4 \times 10^{-5} W)(20 cm/s)}{2 \cdot 5 cm} = 8 \times 10^{-5} W / s$$

Since there are 200 turns of wire, the total induced *emf* is $0.016 W / cm =$

16 mV

6.)

Let d be the height of the slab, and x be the distance from the leftmost edge of the capacitor to the leftmost edge of the dielectric.

a) When the slab is moved within the capacitor, the capacitance changes and a force is exerted on the slab. We start by looking at the differential energy conservation equation:

$$VdQ = d \left(\frac{1}{2} CV^2 \right) + Fdx$$

where C is the capacitance, $V = \frac{Q}{C}$ is the voltage between the plates, Q is the charge on the plates, and F is the force on the slab.

$$0 = CVdV + \frac{V^2}{2}dC + Fdx$$

Using the product rule on the first term on the right hand side of the equation above yields

$$0 = \frac{Q}{C}dQ + Q^2d \left(\frac{1}{C} \right) + \frac{Q^2}{2C^2}dC + Fdx$$
$$0 = -\frac{Q^2}{C^2}dC + \frac{Q^2}{2C^2}dC + Fdx \rightarrow F = \frac{Q^2}{2C} \frac{dC}{dx}$$

The effective capacitance is obtained by considering two capacitors in parallel; one with length equal to x and permittivity ϵ_0 , and one with length equal to $L-x$ with permittivity ϵ .

$$C = \frac{ax\epsilon_0}{d} + \frac{a(L-x)\epsilon}{d} = \frac{ax\epsilon_0 + aL\epsilon - ax\epsilon}{d} = \frac{a(x(\epsilon_0 - \epsilon) + L\epsilon)}{d}$$

where a is the area of each plate in the capacitor.

$$\frac{dC}{dx} = \frac{a(\epsilon_0 - \epsilon)}{d}$$
$$F = \frac{Q^2d}{2a(x(\epsilon_0 - \epsilon) + L\epsilon)} \frac{a(\epsilon_0 - \epsilon)}{d} = \frac{Q^2(\epsilon_0 - \epsilon)}{2(x(\epsilon_0 - \epsilon) + L\epsilon)}$$

Now we rewrite this expression in terms of the dielectric constant $\kappa = \frac{\epsilon}{\epsilon_0}$

$$F = \frac{Q^2(\epsilon_0 - \kappa\epsilon_0)}{2(x(\epsilon_0 - \kappa\epsilon_0) + L\kappa\epsilon_0)} = \boxed{\frac{Q^2(1 - \kappa)}{2(x(1 - \kappa) + L\kappa)}}$$

Since $\kappa > 1$, the force is directed inward. Therefore, to maintain the slab in its position, a force must be exerted *outward*.

b) Again, we start with the differential energy equation, but this time the plates are maintained at a constant potential, V_0 . Thus, $dV=0$, and we have

$$VdQ = CVdV + \frac{V^2}{2}dC + Fdx$$

$$V(CdV + VdC) = CV_0dV + \frac{V_0^2}{2}dC + Fdx$$

$$V_0^2dC = \frac{V_0^2}{2}dC + Fdx \rightarrow F = \frac{V_0^2}{2} \frac{dC}{dx} = \frac{V_0^2}{2} \frac{a(1-\kappa)\epsilon_0}{d}$$

The work that must be done to move the slab to the position $x = \frac{L}{2}$ is

$$W = \int \vec{F} \cdot d\vec{x} = \frac{V_0^2 a \epsilon_0 (1 - \kappa)}{2d} \left(\frac{L}{2} \right) = \boxed{\frac{V_0^2 a L \epsilon_0 (1 - \kappa)}{4d}}$$

8.)

a) We start by taking the curl of Faraday's Law:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{d}{dt} \vec{\nabla} \times \vec{B}$$

The first two terms on the right hand side appear by virtue of a vector product identity. Because there is a zero charge density, Gauss' Law stipulates that the first term on the right hand side is zero. Also, Ampere's Law states:

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon \frac{d\vec{E}}{dt} \right) = \mu_0 \sigma \vec{E} + \mu_0 \epsilon \frac{d\vec{E}}{dt}$$

The second equality is true by virtue of Ohm's Law: $\vec{J} = \sigma \vec{E}$ where σ is the conductivity. Using this result yields:

$$\nabla^2 \vec{E} = \frac{d}{dt} \left(\mu_0 \sigma \vec{E} + \mu_0 \epsilon \frac{d\vec{E}}{dt} \right)$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \sigma \frac{d\vec{E}}{dt} + \mu_0 \epsilon \frac{d^2 \vec{E}}{dt^2}}$$

For an E-field traveling along the z-axis and polarized along the x-axis, the solution is

$$\vec{E} = E_0 \hat{x} e^{i(kz - \omega t)}$$

Plugging this into the equation above yields

$$-k^2 \vec{E} = \vec{E} (\mu_0 \epsilon \omega^2 - i \mu_0 \sigma \omega)$$

$$k^2 = \mu_0 \sigma \omega i - \mu_0 \epsilon \omega^2 \rightarrow k^2 = 4\pi \sigma \mu_0 \omega \left(\frac{i}{4\pi} - \frac{\epsilon \omega}{4\pi \sigma} \right)$$

If $\frac{4\pi\sigma}{\omega\epsilon} \gg 1$ then $k = \sqrt{4\pi\sigma\mu_0\omega} \sqrt{\frac{i}{4\pi}} = \sqrt{\sigma\mu_0\omega i} \rightarrow \infty$.

Thus, waves cannot propagate inside a good conductor at low frequencies. Also, the skin depth is zero and the reflected wave suffers a π phase shift.