Classical Mechanics and Electrodynamics Prelim

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Classical Mechanics Questions

1.

A uniform solid sphere rolls without slipping on a curved track in the vertical plane as shown in the figure. The sphere of radius $r < R$ and mass m is released from rest at height h as shown.

a) The sphere will just make it through the loop if the normal force at the top of the loop (point b) is zero. Newton's 2nd Law at b reads:

$$F_b = \frac{mv_b^2}{R-r} = mg + N_b$$

When $N_b = 0$, we have

$$v_b^2 = (R-r)\,g$$

We can relate this to the height from which the sphere is released by energy conservation considerations of the center of mass:

$$mgh = \frac{1}{2}mv_b^2 + \frac{1}{2}I\omega^2 + mg\,(2R-r)$$

where the right hand side of the equation above represents the total energy of the sphere at point b. Here, $I = \frac{2}{5}mr^2$ represents the moment of inertia of the solid sphere. Since the sphere rolls without slipping, the velocity of the center of mass is related to the angular velocity by: $v_{cm} = \omega r$. Using this, we have:

$$gh = \frac{1}{2}v_b^2 + \frac{1}{5}v_b^2 + g\,(2R-r)$$

$$h = \frac{7}{10g}v_b^2 + 2R - r = \frac{7}{10}\,(R-r) + 2R - r = \boxed{\frac{7}{10}\,(27R - 17r)}$$

b) When $0 < (r - R) \ll R$, $r \approx R$ and we find that $h = R$. This makes sense because in this limit, the sphere doesn't move very much inside the loop. As a result, it doesn't lose a significant amount of energy by virtue of rotation, and therefore it doesn't need as much starting potential energy to get through the loop. It is only appropriate that the starting height be equal to the center of mass of the sphere when in the loop.

2.

A bead of mass $m = 1$ is constrained to move in a vertical plane along a wire shaped according to a curve $z = z(x)$ as shown. The bead moves without friction under the influence of gravity which acts in the $-\hat{z}$ direction with a uniform acceleration of 1.

a) The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$$

but $\dot{z} = \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} = z'\dot{x}$. Thus, the Lagrangian reads

$$\boxed{L = \frac{M\dot{x}^2}{2}\left(1 + z'(x)^2\right) - z(x)}$$

b) We can easily express the kinetic energy in terms of $\dot{s}$ as follows:

$$ds = \sqrt{dx^2 + dz^2}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{z}^2}$$

$$\dot{s}^2 + \dot{x}^2 + \dot{z}^2$$

Thus,

$$\boxed{L = \frac{1}{2}m\dot{s}^2 - z(s)}$$

c) If $z(s) = s^2/2$, then

$$L = \frac{\dot{s}^2}{2} - z(s) = \frac{\dot{s}^2}{2} - \frac{s^2}{2}$$

$$\frac{\partial L}{\partial \dot{s}} = \dot{s} \quad \frac{\partial L}{\partial s} = -s$$

The Euler-Lagrange equation of motion is then:

$$\ddot{s} + s = 0$$

The solution of this equation is $s = Acost + Bsint$. The initial velocity of the particle is zero, which implies that $B = 0$ and therefore, $s = Acost$. Regardless of the starting position, the particle will always reach $s = 0$ in a time $t = \pi/2$ in this interval.

d) We can derive the said differential equation by proceeding as follows

$$\frac{dz}{dx} = \frac{\partial z}{\partial s}\frac{ds}{dx} = s\frac{ds}{dx} = \sqrt{2z}\sqrt{1+\left(\frac{dz}{dx}\right)^2}$$

$$\left(\frac{dz}{dx}\right)^2 = 2z + 2z\left(\frac{dz}{dx}\right)^2$$

$$\left(\frac{dz}{dx}\right)^2 (1-2z) = 2z$$

$$\left(\frac{dz}{dx}\right)^2 = \frac{2z}{1-27} = \left(\frac{1}{2z}-1\right)^{-1}$$

$$\boxed{\frac{dz}{dx} = \pm\left(\frac{1}{2z}-1\right)^{-1/2}}$$

3.

A uniform rectangular block of mass m and dimensions $a \times b \times c$ rests on a horizontal cylinder of radius r as shown in the figure.

a) We start by determining the coordinates of the pivot when the block is perfectly horizontal:

$$x_{piv} = 0 \quad z_{piv} = r$$

We then determine these coordinates when the block is rotated at some angle θ. Notice that the hypotenuse of the right triangle that is formed is $r + b/2$, and not just r.

$$x'_{piv} = -\left(r + \frac{b}{2}\right) sin\theta \quad z'_{piv} = \left(r + \frac{b}{2}\right) cos\theta$$

From this information, we can determine the coordinate of the center of mass of the block. The key here is that there is an additional translation by an amount equal to one of the legs of a right triangle whose hypotenuse is the arc length traced out by the block as it rotates without slipping on the cylinder: $r\theta$.

$$x'_{cm} = -\left(r + \frac{b}{2}\right) sin\theta + r\theta cos\theta \quad z'_{cm} = \left(r + \frac{b}{2}\right) cos\theta + r\theta sin\theta$$

or equivalently,

$$\boxed{x = r\left(\theta cos\theta - sin\theta\right) - \frac{b}{2} sin\theta}$$

$$\boxed{z = r\left(\theta sin\theta + cos\theta\right) + \frac{b}{2} cos\theta}$$

b) We start by writing down the potential energy of the system:

$$U = mgh = mg\left(\left(r + \frac{b}{2}\right) cos\theta + r sin\theta\right)$$

Stable equilibrium is achieved when the derivative of the potential with respect to θ is zero, and the potential is minimized:

$$U' = 0 = mg\left(-\left(r + \frac{b}{2}\right) sin\theta + r sin\theta + r\theta s cos\theta\right)$$

$$r\theta = \frac{b}{2}tan\theta$$

$\theta = 0$ satisfies the expression above and is indeed an equilibrium point. Now we have to make sure that the potential is minimized at $\theta = 0$ by checking that the second derivative is greater than zero: $U'' > 0$

$$U'' = mg\left(-\left(r+\frac{b}{2}\right)cos\theta + rcos\theta + r\theta sin\theta\right)$$

We see that this expression is greater than zero if $\boxed{r - \frac{b}{2} > 0}$. Notice that there are no condition on a or c.

c) The restoring force that the block experiences is

$$F = -\frac{U'}{r'} = mr'\ddot{\theta} = -\frac{mg}{r'}\left(r\theta cos\theta - \frac{b}{2}sin\theta\right)$$

$$r'^2\ddot{\theta} = g\left(\frac{b}{2}sin\theta - r\theta cos\theta\right)$$

For small oscillations, $sin\theta = \theta$ and $cos\theta = 1$. Thus, we have:

$$\ddot{\theta} = \frac{g}{r'^2}\left(\frac{b}{2} - r\right)\theta$$

where $r'^2 = x^2 + z^2$:

$$r'^2 = \left(-\left(r+\frac{b}{2}\right)\theta + r\theta\right)^2 + \left(\left(r+\frac{b}{2}\right) + r\theta^2\right)^2$$

Dropping terms that are second order in θ or higher yields:

$$r'^2 = r^2 + \frac{b^2}{4} + rb = \left(r+\frac{b}{2}\right)^2$$

Using this result in the expression for $\ddot{\theta}$ gives us:

$$\ddot{\theta} = -\frac{g}{\left(r+\frac{b}{2}\right)^2}\left(r - \frac{b}{2}\right)\theta$$

This differential equation represents oscillation in θ:

$$\ddot{\theta} = -\omega^2\theta$$

with

$$\boxed{\omega = \frac{\sqrt{g\left(r - \frac{b}{2}\right)}}{r + \frac{b}{2}}}$$

Note that because $r > \frac{b}{2}$, ω is real, as expected.

4.

A classical body of charge q and mass m is moving under the influence of electromagnetic fields specified by $\phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$.

a) The kinetic and potential energies of the system are

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\phi - q\vec{A} \cdot \vec{v}$$

and thus, the Lagrangian is

$$L - T - U = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\phi + q\vec{A} \cdot (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})$$

The equations of motion can be obtained by taking the derivative of the Lagrangian with respect to the velocity

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x\dot{x} = p_x \rightarrow \dot{x} = \frac{p_x - qA_x}{m}$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y\dot{y} = p_y \rightarrow \dot{y} = \frac{p_y - qA_y}{m}$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z\dot{z} = p_z \rightarrow \dot{z} = \frac{p_z - qA_z}{m}$$

The Hamiltonian is given by

$$H = \sum_i p_i\dot{q}_i - L = p_x\left(\frac{p_x - qA_x}{m}\right) + \ldots - \frac{m}{2}\left(\frac{(p_x - qA_x)^2}{m^2} + \ldots\right) + q\phi - aA_x\frac{(p_x - qA_x)}{m} + \ldots$$

It is important to note that the Hamiltonian must be a function of p and *not* $\dot{q}$.

$$H = \frac{p_x - qA_x}{m}\left(p_x + \ldots - \frac{p_x - qA_x}{2} + \ldots - qA_x\right) + q\phi$$

$$= \frac{p_x - qA_x}{m}\left(\frac{p_x}{2} - \frac{qA_x}{2} + \ldots\right) + q\phi$$

$$= \boxed{\frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m} + \frac{(p_z - qA_z)^2}{2m} + q\phi}$$

b) We proceed by deriving Hamilton's equations of motion:

$$\dot{p_x} = -\frac{\partial H}{\partial \dot{x}} = 0$$

This implies that p_x is a conserved quantity (the same is also true for p_y and p_z).

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x - qA_x}{m}$$

$$m\dot{x} = p_x - qA_x$$

$$\boxed{\vec{p} = m\vec{v} + q\vec{A} \neq m\vec{v}}$$

c) The Lagrangian is given by:

$$\boxed{\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\phi + q\vec{A} \cdot (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})}$$

The principle of least action results in the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

The Euler-Lagrange equations for our system are

$$m\ddot{x} + q\dot{A}_x + q\partial_x\phi = 0$$

$$m\ddot{y} + q\dot{A}_y + q\partial_y\phi = 0$$

$$m\ddot{z} + q\dot{A}_z + q\partial_z\phi = 0$$

Thus, we get the familiar Lorentz force law:

$$m\ddot{\vec{r}} = \vec{F} = q\vec{E}$$

where $\vec{E} = -\vec{\bigtriangledown}\phi - \frac{d\vec{A}}{dt}$. This describes a non-conservative electric field produced by a changing magnetic field and therefore appropriately describes electromagnetic fields.

Electricity and Magnetism Questions

1.

A classical model of an electron consists of a negative charge $-e$ uniformly distributed within a sphere of radius R.

a) The charge density of the sphere is $\rho = -\frac{3e}{4\pi R^3}$. Using Gauss' Law, we find that for $r < R$:

$$E(4\pi r^2) = \frac{\rho}{\epsilon_0}\left(\frac{4}{3}\pi r^3\right)$$

$$\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r} = -\frac{3er}{12\pi R^3\epsilon_0}\hat{r} = \boxed{-\frac{er}{4\pi\epsilon_0 R^3}\hat{r}}$$

For $r > R$, the electric field is that produced by a point charge with magnitude e:

$$\boxed{\vec{E} = -\frac{e}{4\pi\epsilon_0 r^2}\hat{r}}$$

b) The electrostatic potential energy is given by

$$W = \frac{\epsilon_0}{2}\int\left|\vec{E}_0\right|^2 dV = \frac{\epsilon_0}{2}\left[\int_0^r \frac{e^2 r^2}{16\pi^2\epsilon_0^2 R^6} r^2 sin\theta d\theta d\phi dr + \int_R^\infty \frac{e^2}{16\pi^2\epsilon_0^2 r^4} r^2 sin\theta d\theta d\phi dr\right]$$

$$= \frac{\epsilon_0}{2}\left[\frac{e^2}{4\pi\epsilon_0^2 R^6}\int_0^R r^2 dr + \frac{e^2}{4\pi\epsilon_0^2}\int_R^\infty \frac{dr}{r^2}\right]$$

$$= \frac{e^2}{8\pi\epsilon_0}\left[\frac{1}{R^6}\left(\frac{R^5}{5}\right) - \left(0 - \frac{1}{R}\right)\right]$$

$$= \frac{e^2}{8\pi\epsilon_0}\left[\frac{1}{5R} + \frac{1}{R}\right] = \frac{6e^2}{40\pi\epsilon_0 R} = \boxed{\frac{3e^2}{20\epsilon_0\pi R}}$$

c) We proceed by setting W equal to the rest energy of the electron:

$$\frac{3e^2}{20\pi\epsilon_0 R} = mc^2$$

$$R = \frac{3e^2}{20\pi\epsilon_0 mc^2} = \boxed{\frac{3}{5}r_0}$$

where $r_0 = e^2/4\pi\epsilon_0 mc^2$ is the classical electron radius.

2.

A long insulating cylinder of radius a is placed in a constant magnetic field that is uniform through all space so that its axis is perpendicular to the field.

a) We start by writing down the constitutive equation that relates the magnetic field to the auxiliary field:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \rightarrow \vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right)$$

Gauss' Law read:

$$\vec{\nabla} \cdot \vec{B} = 0 = \vec{\nabla} \cdot \vec{H} + \vec{\nabla} \cdot \vec{M}$$

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$$

The magnetization within the cylinder is given by

$$\vec{M} \times \hat{n} = \vec{k_b}$$

where $\hat{n}$ is a unit vector normal to the surface of the cylinder and $\vec{k_b}$ is the bound surface current density. The parallel component of $\vec{k_b}$ is given by

$$\hat{n} \times \vec{k_b} = \hat{n} \times \left(\vec{M} \times \hat{n}\right) = \vec{M}$$

Now we take the divergence of both sides

$$\vec{\nabla} \cdot \vec{M} = \vec{\nabla} \cdot \left(\hat{n} \times \vec{k_b}\right)$$

$\vec{k_b}$ only exists on the surface of the cylinder so the divergence of $\vec{M}$ is zero everywhere except at $\rho = a$. We've already shown that

$$\vec{\nabla} \cdot \vec{M} = -\vec{\nabla} \cdot \vec{H}$$

Therefore,

$$\boxed{\vec{\nabla} \cdot \vec{H} = 0}$$

everywhere except on the surface of the cylinder.

b) Since there is no free current in the problem, Ampere's Law states that $\vec{\bigtriangledown} \times \vec{H} = 0$. Therefore, $\vec{H}$ can be expressed as the gradient of a scalar field:

$$\vec{H} = -\vec{\bigtriangledown}\Phi$$

Since the divergence of $\vec{H}$ is zero, we recover LaPlace's equation:

$$\bigtriangledown^2\Phi = 0$$

The general solution (in cylindrical coordinates) is

$$\Phi = Aln\rho + \sum_m \left(B_m\rho^m sin\left(m\phi\right) + C_m\rho^{-m} sin\left(m\phi\right) + D_m\rho^m cos\left(m\phi\right) + E_m\rho^{-m} cos\left(m\phi\right)\right) + F$$

The boundary conditions are:

$$\text{i) } \Phi(\rho \to 0) \neq \infty$$

This says that the potential cannot blowup at the origin.

$$\text{ii) } \Phi(\rho \to \infty) = -H\rho cos\phi = -\frac{B_0}{\mu_0}\rho cos\phi$$

This says that the potential must approach that for the original field far away from the cylinder. Here, B_0 is the strength of the original magnetic field.

$$\text{iii) } H_O^{\parallel} - H_I^{\parallel} = 0$$

This is true because there is no free current on the surface of the cylinder.

$$\text{iv) } B_O^{\perp} - B_I^{\perp} = 0$$

This is always true. Note that subscripts I and O denote the field inside and outside the cylinder respectively. Using BC (i), we find that

$$\Phi_I = F + \sum_m \left(B_m\rho^m sin\left(m\phi\right) + D_m\rho^m cos\left(m\phi\right)\right)$$

BC (ii) gives us

$$\Phi_O = \sum_m \left(G_m\rho^m sin\left(m\phi\right) + H_m\rho^m cos\left(m\phi\right)\right) = -\frac{B_0}{\mu_0} cos\phi$$

This implies that $G_m = 0$ and $H_1 = -B_0/\mu_0$. Thus,

$$\Phi_O = -\frac{B_0}{\mu_0}\rho cos\phi + \sum_m \left(E_m\rho^{-m} sin\left(m\phi\right) + F_m\rho^{-m} cos\left(m\phi\right)\right)$$

Since $H^{\|} = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi$, BC (iii) yields:

$$\sum_m \left(mB_m a^m cos\left(m\phi\right) - mD_m a^m sin\left(m\phi\right)\right) = \frac{B_0}{\mu_0} a sin\phi + \sum_m (mE_m a^{-m} cos\left(m\phi\right) - mF_m a^{-m} sin\left(m\phi\right))$$

For $m = 1$,

$$-D_1 a = \frac{B_0 a}{\mu_0} - F_1 a^{-1} \rightarrow F_1 = D_1 a^2 + \frac{B_0 a^2}{\mu_0}$$

For $m \neq 1$,

$$D_m a^m = F_m a^{-m} \quad B_m a^m = E_m a^{-m}$$

Since $B^{\perp} = -\mu\frac{\partial}{\partial\rho}\Phi$, BC (iv) gives us

$$\mu_0\left[= \frac{B_0}{\mu_0}cos\phi + \sum_m \left(-mE_m a^{-m-1} sin\left(m\phi\right) - mF_m a^{-m-1} cos\left(m\phi\right)\right)\right]$$

$$= \mu\left[\sum_m mB_m a^{m-1} sin\left(m\phi\right) + mD_m a^{m-1} cos\left(m\phi\right)\right]$$

For $m = 1$,

$$-B_0 - \mu_0 F_1 a^{-2} = \mu D_1$$

For $m \neq 1$,

$$-m\nu_0 F_m a^{-m-1} = m\mu D_m a^{m-1} \quad -\mu_0 E_m a^{-m-1} = \mu B_m a^{m-1}$$

$$-B_0 - \mu_0\left(D_1 + \frac{B_0}{\mu_0}\right) = \mu D_1$$

$$-2B_0 - \mu_0 D_1 = \mu D_1$$

$$D_1 = \frac{-2B_0}{\mu_0 + \mu}$$

From the other equations, we see that all other terms are zero. Thus,

$$\Phi_I = -2\frac{B_0}{\mu + \mu_0}\rho cos\phi$$

Interestingly enough, this system behaves as if it is characterized by an effective permeability that is the average of the permeability of the cylinder and the vacuum permeability:

$$\Phi_I = -\frac{B_0}{\mu'} \rho cos\phi$$

with $\mu' = \frac{\mu_0 + \mu}{2}$.

$$\vec{H} = -\vec{\bigtriangledown}\Phi = -\left[-\frac{2B_0}{\mu+\mu_0} cos\phi\hat{\rho} + \frac{2B_0}{\mu+\mu_0} sin\phi\hat{\phi}\right] = \frac{2B_0}{\mu+\mu_0}\left[cos\phi\hat{\rho} - sin\phi\hat{\phi}\right]$$

$$\boxed{\vec{B}_I = \frac{2B_0\mu}{\mu+\mu_0}\left(cos\phi\hat{\rho} - sin\phi\hat{\phi}\right)}$$

Notice that this field points in the $\hat{x}$ direction and it doesn't blow up at $\rho = 0$.

3.

A particle of charge q and mass m moves in the xy plane under the influence of a magnetic field $\vec{B} = B_0\hat{z}$.

a) The force on the particle is given by the Lorentz force law:

$$\vec{F} = q\vec{v} \times \vec{B}$$

This will lead to a circular orbit such that the sum of the forces is equal to the centrifugal force.

$$F_{cent} = \frac{mv^2}{r}$$

In cylindrical coordinates (ρ, ϕ, and z) we have:

$$-\frac{qp_0 B_0}{m} = -\frac{p_0^2}{m\rho}$$

$$\boxed{\rho(t) = \frac{p_0}{qB_0}}$$

The cyclotron frequency is given by:

$$\omega_{cyc} = \frac{2\pi}{T} = \frac{2\pi v}{2\pi\rho} = \frac{p_0}{m\rho} = \frac{qB_0}{m}$$

b) Now we add a small transverse electric field $\vec{E} = \epsilon\hat{x}$. We shall proceed by transforming the fields to a frame that is moving at a constant velocity in the plane transverse to the magnetic field, such that $E = 0$. In other words, we'll choose a velocity such that the Lorentz transform of the electric field in the new frame is zero:

$$E'_x = 0 = \gamma\left(-E_x - vB_z\right)$$

The velocity that achieves this transformation is:

$$\vec{v} = -\frac{E_x}{B_z}\hat{y} = \frac{-\epsilon}{B_0}\hat{y}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Since the magnetic field is oriented along the z direction, $E'_y = E'_z = 0$. The magnetic field transforms as:

$$B'_z = \gamma\left(B_z + \frac{v}{c^2}E_x\right) = \gamma\left(B_z - \frac{\epsilon^2}{B_0 c^2}\right)$$

Since the electric field is small, the ϵ^2 term is negligible. Also, since $\epsilon \ll B_0$, the velocity of the co-moving frame is small and therefore, $\gamma \approx 1$. Thus, we have $B'_z \approx B_z$. Since the electric field is oriented in the x direction, $B'_y = B'_x = 0$. In a frame that is rotating with the particle (at the cyclotron frequency), we have uniform circular motion with

$$\vec{\rho}' = \frac{p_0}{qB_0}\left(cos\phi\hat{x} + sin\phi\hat{y}\right) = \frac{p_0}{qB_0}\left(sin\left(\frac{qB_0t}{m}\right)\hat{x} + cos\left(\frac{qB_0t}{m}\right)\hat{y}\right)$$

where the prime refers to the ρ coordinate in the rotating frame. Transforming back to the stationary frame yields

$$\vec{\rho}(t) = \vec{v}t + \vec{\rho}' = \boxed{\frac{p_0}{qB_0}\left(sin\left(\frac{qB_0t}{m}\right)\hat{x} + \left(cos\left(\frac{qB_0t}{m}\right) - \frac{\epsilon}{B_0}t\right)\hat{y}\right)}$$

4.

We are given the following information:

$$\vec{r}(t) = \frac{1}{2}\left[\vec{\rho}e^{-i\omega t} + \vec{\rho}^{*}e^{i\omega t}\right]$$

$$\vec{d}(t) = \frac{1}{2}\left[\vec{\delta}e^{-i\omega t} + \vec{\delta}^{*}e^{i\omega t}\right]$$

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left[\vec{\varepsilon}(\vec{r})e^{-i\omega t} + \vec{\varepsilon}^{*}(\vec{r})e^{i\omega t}\right]$$

a) The electric field in the neighborhood of the charged particles is approximated as

$$\vec{\varepsilon}(\vec{R}+\vec{r}^{(n)}) \simeq \vec{\varepsilon}(\vec{R}) + r_j^{(n)}\frac{\partial}{\partial R_j}\vec{\varepsilon}(\vec{R})$$

The force exerted on an electric dipole due to an electric field is

$$\vec{F} = \left(\vec{d}\cdot\vec{\nabla}\right)\vec{E}$$

$$\begin{aligned}\vec{d}\cdot\vec{\nabla} &= \frac{1}{2}\left(e^{-i\omega t}\left(\delta_x\partial_x + \delta_y\partial_y + \delta_z\partial_z\right) + e^{i\omega t}\left(\delta_x^*\partial_x + \delta_y^*\partial_y + \delta_z^*\partial_z\right)\right)\\ &= \frac{1}{2}\left[\left(e^{-i\omega t}\delta_x + e^{i\omega t}\delta_x^*\right)\partial_x + \left(e^{-i\omega t}\delta_y + e^{i\omega t}\delta_y^*\right)\partial_y + \ldots\right]\end{aligned}$$

$$\left(\vec{d}\cdot\vec{\nabla}\right)\vec{E} = \frac{1}{4}\left(e^{-2i\omega t}\delta_x\frac{\partial}{\partial R_x}\vec{\varepsilon}(\vec{R}) + \delta_x\frac{\partial}{\partial R_x}\vec{\varepsilon}^{*}(\vec{R}) + \delta_x^*\frac{\partial}{\partial R_x}\vec{\varepsilon}(\vec{R}) + \delta_x^*\frac{\partial}{\partial R_x}\vec{\varepsilon}^{*}(\vec{R})e^{2i\omega t} + \ldots\right]$$

The time-averaged force is given by

$$\langle F\rangle = \frac{1}{T}\int_0^T F dt$$

$$= \frac{1}{4T}[\left(\vec{\delta}\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R})\left(\frac{-1}{2i\omega}\right)\left(e^{-2i\omega T}-1\right) + \left(\vec{\delta}\cdot\vec{\nabla}\right)\vec{\varepsilon}^{*}(\vec{R})T + \left(\vec{\delta}^{*}\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R})T + \left(\vec{\delta}^{*}\cdot\vec{\nabla}\right)\vec{\varepsilon}^{*}(\vec{R})\left(\frac{1}{2i\omega}\right)\left(e^{2i\omega T}-1\right)]$$

$$= \frac{1}{4}\left[\left(\vec{\delta}\cdot\vec{\nabla}\right)\vec{\varepsilon}^{*}(\vec{R}) + \left(\vec{\delta}^{*}\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R})\right]$$

$$= \boxed{\frac{1}{2}Re\left[\left(\vec{\delta}^{*}\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R})\right]}$$

b) Farraday's Law for this system yields:

$$\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt} = \hat{x}\frac{1}{2}\left[\left(\frac{\partial}{\partial R_y}\varepsilon_z(\vec{R}) - \frac{\partial}{\partial R_z}\varepsilon_y(\vec{R})\right)e^{-i\omega t} + \left(\frac{\partial}{\partial R_y}\varepsilon_z^*(\vec{R}) - \frac{\partial}{\partial R_z}\varepsilon_y^*(\vec{R})\right)e^{i\omega t}\right] + \dots$$

$$= \hat{x}\left(\frac{\partial}{\partial R_y}Re\left(\varepsilon_z(\vec{R})e^{-i\omega t}\right) - \frac{\partial}{\partial R_z}Re\left(\varepsilon_y(\vec{R})e^{-i\omega t}\right)\right] + \dots$$

$$= \hat{x}Re\left[\frac{1}{i\omega}e^{-i\omega t}\left(\frac{\partial}{\partial R_y}\varepsilon_z - \frac{\partial}{\partial R_z}\varepsilon_y\right)\right] + \dots = Re\left[\frac{e^{i\omega t}}{i\omega}\vec{\nabla}\times\vec{\varepsilon}(\vec{R})\right]$$

This shows that

$$\boxed{\vec{B}(\vec{R}) = Re\left[\frac{\vec{\nabla}\times\vec{\varepsilon}(\vec{R})}{i\omega}\right]}$$

which is the answer to part (c). The Lorentz force is given by:

$$F^{(n)} = q^{(n)}\dot{\vec{r}}^{(n)}\times\vec{B}$$

where $\dot{\vec{r}}^{(n)} = Re\left(-i\omega\vec{\rho}^{(n)}e^{-i\omega t}\right)$.

$$F = \sum_n Re\left(q^{(n)}(-i\omega)\rho^{(n)}e^{-i\omega t}\times\frac{e^{-i\omega t}}{i\omega}\vec{\nabla}\times\vec{\varepsilon}(\vec{R})\right)$$

$$= -Re\left[\vec{\delta}^*e^{i\omega t}\times e^{-i\omega t}\vec{\nabla}\times\vec{\varepsilon}(\vec{R})\right] = \boxed{Re\left[\vec{\delta}^*\times i\omega\vec{B}(\vec{R})\right]}$$

d) The total cycle-averaged force on the dipole is given by

$$\vec{F} = \vec{F}_E + \vec{F}_B = \frac{1}{2}\left[\left(\vec{\delta}^*\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R})\right] + Re\left[\vec{\delta}^*\times\left(\vec{\nabla}\times\vec{\epsilon}(\vec{R})\right)\right]$$

$$= \frac{1}{2}Re\left[\left(\vec{\delta}^*\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R}) + \vec{\nabla}\left(\vec{\delta}^*\cdot\vec{\varepsilon}(\vec{R})\right) - \left(\vec{\delta}^*\cdot\vec{\nabla}\right)\vec{\varepsilon}(\vec{R})\right]$$

$$= \frac{1}{2}Re\left[\vec{\nabla}\left(\vec{\delta}^*\cdot\vec{\varepsilon}(\vec{R})\right)\right]$$

In index form, this reads:

$$\boxed{F_i = \frac{1}{2}Re\left[\delta_j^*\frac{\partial}{\partial R_i}\varepsilon_j(\vec{R})\right]}$$