Physics 151

Exam 1

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Instructions:

- Answer three questions from section 1 and the long question in section 2.
- You are permitted to use a calculator.
- You should neglect air resistance and wind unless otherwise stated.
- Show all your working for credit.
- Draw a box around your final solution to each question.
- Make your work presentable and clear to follow. You could receive more credit this way.
- Time allowed: 2 hours.
- On your blue book write your name and ID (peoplesoft) number and lab section along with the version of the exam you took.
- You may keep the exam paper, the formula sheet and scrap paper if you wish. Return your blue book to me.
- Good Luck!

1. **Projectile Motion**
You throw a brick with an intial speed u at an angle θ to the (level) ground.

(a) What is the angle in radians required such that the ball remains in the air the longest?

(b) What value of θ (in radians) gives the greatest height?

(c) What value of θ (in radians) gives the greatest range?

Make sure that you justify your answers mathematically.

Solution:

(a) We need to find the time of flight and then maximise it. First, let's analyse the problem. We're dealing with an object moving only under the influence of its weight, so its acceleration will be $\mathbf{a} = -g\hat{\jmath}$. The problem is therefore defined by

$$\ddot{x} = 0 \qquad\qquad \ddot{y} = -g \tag{1}$$

where the x-axis lies along the ground and the y-axis is vertical. The initial conditions are $\dot{y}(0) = u \sin\theta$ and $\dot{x}(0) = u\cos\theta$
Integrating the y equation once with the initial condition $\dot{y}(0) = u\sin\theta$ we obtain the expression for the y-component of the velocity:

$$\dot{y}(t) = u\sin\theta - gt \tag{2}$$

When the brick reaches its maximum height, the y-component (although not the x-component) of its velocity must be zero. This will occur when $t = T/2$ where T is the time for the whole flight. Our equation thus reduces to

$$0 = u\sin\theta - g\frac{T}{2} \tag{3}$$

$$T = \frac{2u\sin\theta}{g} \tag{4}$$

This expression is a maximum when $\sin\theta = 1 \implies \theta = \frac{\pi}{2}$, i.e. straight up.

(b) Similarly, we now need an expression for the height in terms of the fixed parameters. Remember that for constant acceleration we had the expression $v^2 = u^2 + 2as$. In our notation this is the same as $\dot{y}^2(t) = \dot{y}^2(0) - 2g\Delta y$ where $\Delta y = y(t) - y(0)$. We could also get this equation directly from (1) by using the chain rule:

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{y}}{dy} \tag{5}$$

Equation (1) for y then becomes

$$\dot{y}\frac{d\dot{y}}{dy} = -g \tag{6}$$

which integrates to give

$$\frac{1}{2}\dot{y}^2(t) = -gy(t) + const. \tag{7}$$

Given the initial condition, the constant must be $\frac{1}{2}\dot{y}^2(0) + gy(0)$ and so the equation can be rearranged as

$$\dot{y}^2(t) = \dot{y}^2(0) - 2g\Delta y \tag{8}$$

Given that when the brick reaches its maximum height $\Delta y = h$, we have

$$\dot{y}^2(t) = \dot{y}^2(0) - 2gh \tag{9}$$

At this point, as in part (a), we know that $\dot{y}(t) = 0$. Also, $\dot{y}(0) = u\sin\theta$ and so

$$0 = u^2\sin^2\theta - 2gh \tag{10}$$

$$h = \frac{u^2\sin^2\theta}{2g} \tag{11}$$

This expression also maximises when $\sin\theta = 1 \implies \theta = \frac{\pi}{2}$, i.e. straight up.

(c) The range R is the same as the x component of displacement for the whole flight. We already have an expression for the time of flight,

$$T = \frac{2u\sin\theta}{g} \tag{12}$$

and, integrating the x equation from (1) twice while applying the initial conditions

$$x(0) = 0,\ \dot{x}(0) = u\cos\theta \tag{13}$$

we have

$$x(t) = ut\cos\theta \tag{14}$$

Combining these two equations we find that

$$x(T) = R = \frac{2u^2\cos\theta\sin\theta}{g} = \frac{u^2\sin 2\theta}{g} \tag{15}$$

Clearly this function maximises when $\sin 2\theta = 1 \implies \theta = \frac{\pi}{4}$

A note regarding notation: remember that the first derivative of position is velocity, i.e. $\dot{y}(t) = v_y(t)$ (similarly for x). Also, the components of the (constant) initial velocity were written as $\dot{y}(0)$ and $\dot{x}(0)$. These could also have been written $v_y(0)$, v_{y0} or u_y (similarly for x) depending on which book you read.

Note also that the final expressions all depend on the parameters of the problem, θ, u and g.

2. For no particular reason, a brick of mass m is hanging vertically from a rope attached to the ceiling.
   (a) Draw a free body diagram showing the forces acting *on* the brick.
   (b) What forces are exerted *by* the brick and *on* what do they act?
   (c) Identify all the third law pairs from the forces idetified in parts (a) and (b).
   (d) Which forces cancel each other out?

Solution:

(a) See diagram

(b) The object exerts a force T *on* the rope and a force mg *on* the Earth, where g is the gravitaional field strength of the Earth's field.

(c) i. The weight of the brick, i.e. the gravitational force exerted *by* the Earth *on* the brick has as its third law partner the gravitational force exerted *by* the brick back *on* the Earth, i.e. the weight of the Earth in the gravitational field of the brick.

ii. The force T exerted *by* the brick *on* the rope is paired with the force T exerted *by* the rope *on* the brick.

In each case the paired forces are equal in magnitude and opposite in direction.

(d) The third law pairs do not cancel out, as they don't act on the same object. However, as the system is not accelerating (as it would be were it in a lift, for example), the vertical forces acting *on* each part of the system must cancel out by N2.

i. For the brick, the upward force T exerted *on* it *by* the rope must cancel with the weight of the brick (the force exerted *on* it *by* the Earth). So $T = mg$.

ii. For the rope, the upward force exerted *on* it *by* the ceiling must cancel the downward force exerted *on* it *by* the brick. They are both denoted T.

3. Imagine you are swinging a brick of mass m attached to a string around in a circle above your head. The plane of the circle is parallel to the ground. The tension in the string is T and the radius of the path is r.
   (a) What is the speed of the brick the moment the string breaks?
   (b) Describe the trajectory of the brick after the string breaks according to Newton's laws. Ignore gravity.

Solution:

(a) For the brick to be in circular motion, $T = mv^2/r$. The speed of the brick is therefore $v = \sqrt{\frac{rT}{m}}$

(b) Without gravity, the brick will continue in a straight line with the speed given in part (a).

4. Two bricks are connected as shown over a solid wedge in the shape of a right triangle. The angle which the slope makes with the flat surface is α. The pulley is smooth and massless, the surface is rough with coefficient μ. The string is inextensible. The system is set in motion and brick 1 falls downwards.

   The wedge has been dutifully nailed to a fixed table.
   (a) Draw a diagram showing all forces *on* the bricks.
   (b) Write Newton's second law for the bricks.
   (c) Eliminate the tension in the string to find the acceleration of the bricks in terms of m_1, m_2, μ, g and α

Solution:

(a) See diagram

(b) N2 for brick 1

$$m_1 a = m_1 g - T$$

and for brick 2 perpendicular to the slope

$$0 = m_2 g \cos\theta - N$$
$$\therefore N = m_2 g \cos\theta$$

and parallel to the slope

$$m_2 a = T - m_2 g \sin\theta - \mu N$$

(c) Combining the results for m_2

$$m_2 a = T - m_2 g \sin\theta - \mu m_2 g \cos\theta$$

eliminating T we obtain

$$\begin{aligned} m_2 a &= m_1(g - a) - m_2 g \sin\theta - \mu m_2 g \cos\theta \\ (m_1 + m_2)a &= m_1 g - m_2 g \sin\theta - \mu m_2 g \cos\theta \\ a &= \frac{m_1 g - m_2 g(\sin\theta + \mu\cos\theta)}{m_1 + m_2} \end{aligned}$$

Note that if $m_1/m_2 \to \infty$, $a \to g$, i.e. m_1 experiences freefall if the other brick is negligibly small. Similarly, if $m_1/m_2 \to 0$, $a \to g(\sin\theta - \mu\cos\theta)$ which one would expect if brick 2 were to slide down the slope alone. The frictionless case is, of course, recovered by setting $\mu \to 0$. If we set $\theta = 0$, then we have

$$a = \frac{m_1 g - \mu m_2 g}{m_1 + m_2}$$

which is the result one would expect for the case of a flat table.

It is generally a good idea to check the result in the extreme cases as discussed above and see how it compares to expected results. This is a good way to pick up lost minus signs and factors.

Note also that the final result has been written in terms of the parameters given: m_1, m_2, μ, θ and g and extra variables which were used during the calulation, T and N, have been eliminated.

It is also a good idea to check that the final result is dimensionally correct, i.e. that both sides have the same dimensions. In this case they both have the dimensions of acceleration, $\mathrm{L/T^2}$.